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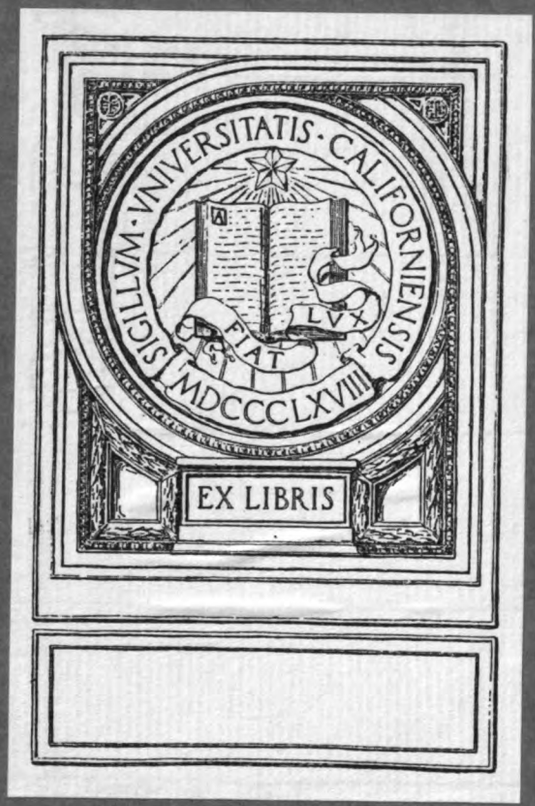

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VOL. XXI, 1929

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I.

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ON A PRIMARY STANDARD OF MUTUAL INDUCTANCE

FOR PRESENTATION TO THE

IMPERIAL GOVERNMENT OF JAPAN

BY

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ABSTRACT.

This paper describes the construction and calculation of a new primary standard of mutual inductance for presentation to the Imperial Government of Japan. It is of the CAMBPELL type, and is very similar to the National Physical Laboratory standard constructed in 1907. A number of improvements have, however, been made in the details of design and construction. In particular the linear dimensions of the coils have been increased in such a manner as to reduce the errors associated with uncertainties in the position and diameter of the secondary coil. The value of the mutual inductance is approximately 10 millihenries (the same as that of the National Physical Laboratory standard).

The measurement of the linear dimensions of the coils is described, and tables and curves of results are given, showing the degree to which the ideal of perfectly uniform helical windings has been approached in the case of the primary coils.

The calculation of the mutual inductance, assuming a perfectly constructed primary winding and a secondary winding of infinitesimally small cross-section, was made by means of the J. VIRIAMU JONES formula, the elliptic integrals being calculated by the methods of L. V. KING. Corrections were then made for irregularities in the pitch and diameter of the primary winding, and also for the finite cross-section of the secondary coil, and thus the final value was obtained.

An experimental comparison of the new standard and the National Physical Laboratory standard was then made, and it is shown that the units realised by the two standards differ from one another by a few parts in a million only.

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ON A PRIMARY STANDARD OF MUTUAL INDUCTANCE FOR PRESENTATION TO THE IMPERIAL GOVERNMENT OF JAPAN.

By D. W. Dye, D.Sc., A.C.G.I., Principal Assistant, and L. Hartshorn, B.Sc., A.R.C.S., D.I.C., Assistant, in the Electricity Department.

[May, 1927.]

I. Introduction.

The Standard of Mutual Inductance of ten millihenries here described has been constructed at the National Physical Laboratory for presentation by the Lord President of His Britannic Majesty's Privy Council, President of the Committee of Council for Scientific and Industrial Research, to the Imperial Japanese Government, as a token of the sympathy of His Majesty's Government for the grievous losses sustained in the earthquake of 1923, in which its predecessor was destroyed.

This former Japanese standard of mutual inductance was designed in accordance with the principles developed at the Laboratory by Mr. A. Campbell,* and has been described by D. W. Dye.† The design and construction of the new standard follow closely those of the old. Some important improvements have, however, been made, partly as the result of previous experience, and partly as the result of a mathematical investigation of the subject by R. M. Wilmotte.‡

The dimensions of the coils recorded in Section 3, p. 5, were measured by Mr. W. H. Johnson, B.Sc., Assistant, and Mr. L. O. C. Johnson, Observer, of the Metrology Department of the Laboratory.

The finished standard has shown itself to be very satisfactory in every way. Comparison measurements with the National Physical Laboratory standard have shown agreement to a very high order of accuracy between the calculated and measured values. Before proceeding to the fuller discussion of the calculations and measurements, a short résumé of the features of design are included here for the sake of completeness.

The standard consists of a primary winding and a secondary winding. It is an accepted fact that the most satisfactory construction of the essentially electrical parts of a primary electrical standard should consist of single-layer coils of bare wire wound under tension on marble cylinders in a screw-cut helical groove.

* A. Campbell, 'N.P.L. Collected Researches', Vol. 4, Paper 6, p. 213, or 'Proc. Roy. Soc.', A, Vol. 79, p. 428, 1907.

† D. W. Dye, 'N.P.L. Collected Researches', Vol. 18, Paper 7, p. 123, or 'Proc. Roy. Soc.', A, Vol. 101, p. 315, 1922.

‡ R. M. Wilmotte, 'Proc. Roy. Soc.', A, Vol. 107, p. 716, 1925.

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In the case of a standard of mutual inductance it is desirable to have a value not less than a few millihenries for convenience. From various considerations this necessitates one of the windings being of many turns, so that it cannot conveniently be in the form of a single-layer helix of bare wire. These considerations led A. CAMPBELL to the design of a standard of mutual inductance of such form that the responsibility for accuracy rested almost entirely upon the accuracy of dimensions of one winding—called for convenience the primary winding—and hardly at all upon the form and dimensions of the other (secondary winding); the latter may, therefore, be overwound.

This is achieved as follows: The primary winding consists of two single-layer helices wound upon the same marble cylinder. These two helices are separated by an interval about twice the length of each. When a current is sent through these windings in series, connected in the same sense, a little consideration will show that in a plane midway between the two helices, and perpendicular to the axis, there is a circle of a certain diameter such that, at its circumference, the magnetic field, both axially and radially, is zero. When, therefore, the secondary winding is concentrated on to this circle, the mutual inductance depends only to a very small extent upon its diameter. Actually, of course, the secondary occupies a finite space, and small corrections due to this are necessary. The secondary winding actually consists of a multi-layer coil wound within a space about 1 cm. wide and 1 cm. deep upon a marble ring about 50 cm. in diameter.

II. CONSTRUCTION.

The main constructional features of the standard can be seen from the accompanying photograph, Fig. 1. A circular slab of marble about 60 cm. in diameter is supported on three hard ebonite levelling screws working in ebonite threaded bushes. The base has a central shallow recess on its upper surface into which the primary marble cylinder fits without side play. The base also carries three rectangular marble pillars held down by hard ebonite bolts. These pillars are fitted with ebonite levelling screw supports at their upper ends, on which the marble ring carrying the secondary winding sits. Three centring screws on the pillars serve to adjust and retain the marble ring in its position concentric with the primary winding.

The upper flat surfaces of the primary cylinder and the secondary ring have been machined each at the same set-up in the lathe at which the grooves for the windings were cut, so that the plane of each may be considered accurately perpendicular to the axis.

The process of adjustment, therefore, consists in first levelling the primary cylinder by means of the three main levelling screws on the base, using a good spirit level in two positions on the upper flat surface of the primary cylinder. The secondary ring is then levelled, using the levelling screws on the marble pillars. The two axes are then vertical and therefore parallel.

Each primary winding consists of 100 turns of hard drawn non-magnetic copper

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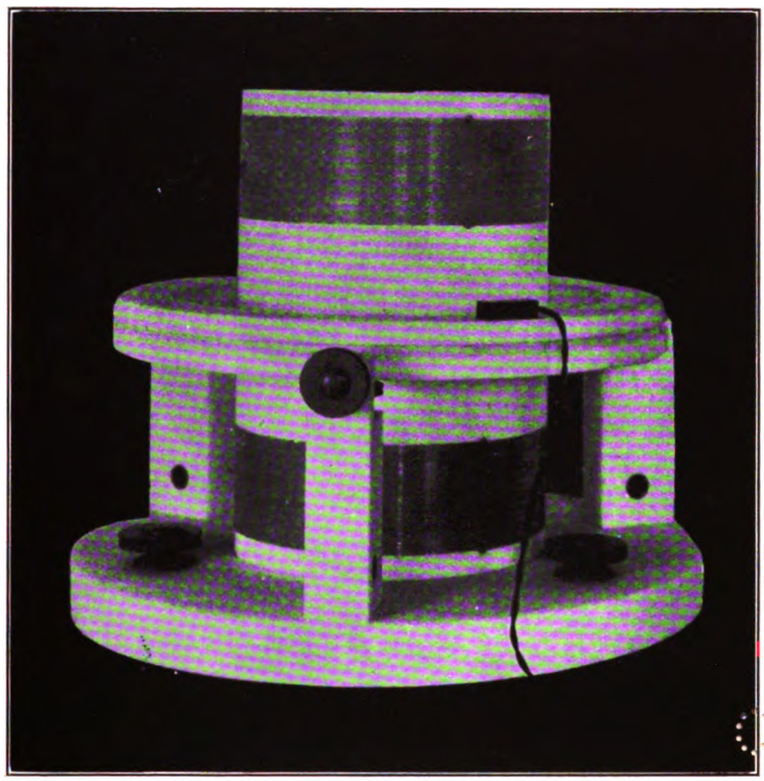

FIG. 1.

[*To face p.* 4.

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wire wound under tension on to the screw-cut groove of 1 mm. pitch on the marble cylinder. The ends are fastened to tapered plugs fitting tightly into the marble. The inner ends of these plugs have thin wire leads soldered to them, to bring the ends of the windings up to terminal sockets on the top face of the cylinder. The loops formed by these leads are disposed so as to have zero mutual inductance to the secondary winding. Two-pin plugs with flexible leads to a terminal board complete the primary windings. The reversal of one of the plugs in its socket enables the upper and lower windings to be connected in opposition when adjusting the height of the secondary winding to its position of electrical symmetry between the two primary helices.

The secondary winding consists of 465 turns wound in 22 layers of 21 turns each and an additional 3 turns on the outside. The wire is double silk covered, of outside diameter 0·0477 cm., the diameter of the bare wire being 0·0411 cm.

Preliminary calculations and experiments with the wire enable a fairly accurate estimate to be made of the inside diameter of the innermost layer. The value thus obtained was 25·2 cm. and the groove in the secondary ring was machined to this diameter. The diameters of the successive layers were then calculated. During the winding process layers of tissue paper were interposed between the layers of the winding. The number of layers of paper was chosen so that each layer of wire had a diameter within about 0·02 mm. of the calculated value.

The final value of the mean radius was 25·712 cm., which, within experimental errors, is identical with the calculated radius of the neutral circle, 25·7128 cm.

The additional three turns were wound on afterwards in order to bring the value of the mutual inductance to within one part in a thousand of 10 millihenries. The ends of the winding are brought by means of flexible leads to a small separate terminal board.

III. Measurement of the Pitch and Mean Diameters of the Primary Windings.

Each coil consists of 100 complete turns of wire having a nominal pitch of 1 mm. and a diameter of about 30 cm. The distance from the centre of one coil to the centre of the other coil is nominally 30 cm.

One end face of the cylinder is scribed with radial lines spaced $22\frac{1}{2}^\circ$ apart, these being numbered in a clockwise direction from 1 to 16. Corresponding generating lines (or generators) on the surface of the cylinder are identified in the same way. A further radial line half-way between lines 1 and 16 has also been marked, and the coil nearer the scribed face of the cylinder commences and ends on the generator corresponding to this extra radial line. The other coil commences and ends on a generator about $7\frac{1}{2}^\circ$ distant from this, but nearer to generator 16. These coils are referred to respectively as B and A.

All measurements have been made commencing at the end of the cylinder remote from the end scribed with the radial lines, the turns of the coils being numbered in the same sense from 1 to 200.

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Thermal Coefficient of Expansion of the Cylinder.—The thermal coefficient of expansion of the cylinder has been determined by making measurements of the variation with temperature in the mean diameter of the coils at twelve selected points, and also by measuring the variation with temperature in the mean distance between the first five turns of coil A and the last five turns of coil B. The various temperatures at which these measurements were made have been attained by varying the heating conditions of the room where the observations were made, low temperatures by flooding the room with cold night and early morning air, and high temperatures by increasing the output of the heating apparatus.

The first series of observations resulted in considerably different values of the coefficient in the two directions, axial and diametral, that for the former being about twice that for the latter. This was an unexpected result and not in accordance with corresponding results obtained for similar cylinders, where the values in the two directions have been found to be closely alike. It was deemed advisable, therefore, to repeat the determinations. This second series of observations was more extensive than the first, the number of observations being approximately three times as great. The values obtained, however, confirmed the existence of the difference previously observed.

Combining the two series of observations, the mean thermal coefficient of expansion of the cylinder for each direction was found to be as follows :—

TABLE I.

Direction.	Mean thermal coefficient of expansion of marble cylinder.	Temperature range.
Axial	0·0000065 per 1° C.	13° C. to 31° C.
Diametral	0·0000035 per 1° C.	13° C. to 33° C.

In searching for a possible explanation for the difference in these two values it was thought that the pressure exerted by the cheeks and mandrel used for supporting the cylinder during the machining and winding, and also during the measurements, might set up strains in the marble that would affect the thermal coefficient of expansion, but subsequent experiments showed that any effect of this kind was negligible.

Diameter of the Coils.—Diameters have been measured by means of the Armstrong-Whitworth large diameter machine fitted with two micrometer measuring heads, using a 300-mm. Hartmann spherical-ended steel gauge as a standard. Readings on the standard were taken immediately before and after a set of observations on the coils.

Measurements of diameter have been made on each turn of the coils at points along opposite generators, at intervals of 45° round the cylinder. These generators, reckoning from one side only, are numbers 1, 3, 5 and 7. The measurements have been necessarily

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made along the rake of the helix, and the observed results corrected accordingly by the amount $-p^2/8D$, where D is the diameter of the coil (300 mm. approximately) and p the pitch of the coil (1 mm.), the correction thus being $-0{\cdot}4\ \mu$. All results have been reduced to 18° C., using the coefficient of expansion for the diametral direction shown in Table I.

The results have been summarised in two ways. First, the mean value of the four measurements taken on each turn of the coils has been found, thus determining the taper of the coils. The resulting mean values are set out in Table II, and are also shown graphically in Fig. 2. Secondly, the mean value of the measurements made along each generator has also been found, thus determining the roundness of the coils. These results are set out in Table III.

The diameter of the wire was measured at 201 points on each coil while it was under tension during the process of winding the wire on the cylinder, and was found to vary between the extremes 0·612 mm. to 0·630 mm. The mean diameter of the wire for each coil has been found to be 0·6194 mm.

The values of the diameters of the coils given in Tables II and III are those found by measuring over the turns of the coils, and the diameters as determined by the axis of the wire are found by allowing for the thickness of the wire. Hence the final mean diameters so determined are as follows :—

Coil A	300·0015 mm.
Coil B	300·0082 mm.
Mean of Coils A and B	300·0048 mm.

Measurement of the Axial Lengths and Spacing of the Turns of the Coils.—These have been measured by means of the Armstrong-Whitworth pitch measuring machine, a micrometer microscope mounted horizontally being employed for viewing each wire in turn and for reading the Laboratory line standard No. 27, whereby the screw of the machine was calibrated. This calibration was carried out immediately before and after a set of observations on the coils, and was confined only to those portions of the screw used in making the measurements of the coils.

The spacing of the turns of the coils has been measured along four generators, numbers 3, 7, 11 and 15, *i.e.*, at intervals of 90° round the cylinder. All the measurements on the coils have been reckoned as from the first turn of A, which for this purpose has been assumed to be a perfect helix, though no measurements have been made to verify this.* All results have been reduced to 18° C., using the coefficient of expansion for the axial direction shown in Table I.

The nominal pitch or spacing of successive turns of each coil is 1 mm., and the nominal distance between the coils, *i.e.*, the distance between corresponding points on the coils

* See ' Note on the First Turn of Coil A', p. 13.

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TABLE II.—Diameters over Wire of Individual Turns of Coils A and B—Taper.

Coil A.				Coil B.			
Turn of coil.	Mean diameter at 18° C.	Turn of coil.	Mean diameter at 18° C.	Turn of coil.	Mean diameter at 18° C.	Turn of coil.	Mean diameter at 18° C.
	mm.		mm.		mm.		mm.
1	300·6087	51	300·6237	101	300·6327	151	300·6308
2	·6047	52	·6242	102	·6327	152	·6390
3	·6070	53	·6201	103	·6279	153	·6324
4	·6050	54	·6250	104	·6292	154	·6305
5	·6086	55	·6265	105	·6315	155	·6268
6	·6066	56	·6274	106	·6332	156	·6251
7	·6079	57	·6282	107	·6364	157	·6298
8	·6096	58	·6286	108	·6346	158	·6229
9	·6094	59	·6250	109	·6365	159	·6300
10	·6111	60	·6270	110	·6344	160	·6256
11	·6121	61	·6241	111	·6357	161	·6252
12	·6162	62	·6237	112	·6357	162	·6281
13	·6165	63	·6262	113	·6352	163	·6291
14	·6187	64	·6248	114	·6330	164	·6250
15	·6157	65	·6298	115	·6368	165	·6264
16	·6155	66	·6208	116	·6321	166	·6199
17	·6184	67	·6206	117	·6247	167	·6257
18	·6173	68	·6219	118	·6296	168	·6171
19	·6221	69	·6213	119	·6215	169	·6196
20	·6172	70	·6225	120	·6261	170	·6185
21	·6209	71	·6228	121	·6323	171	·6218
22	·6201	72	·6203	122	·6303	172	·6208
23	·6184	73	·6173	123	·6342	173	·6211
24	·6225	74	·6227	124	·6315	174	·6232
25	·6218	75	·6249	125	·6346	175	·6259
26	·6239	76	·6231	126	·6370	176	·6234
27	·6220	77	·6251	127	·6380	177	·6216
28	·6239	78	·6227	128	·6333	178	·6183
29	·6223	79	·6217	129	·6347	179	·6154
30	·6252	80	·6247	130	·6336	180	·6160
31	·6240	81	·6239	131	·6387	181	·6174
32	·6262	82	·6242	132	·6359	182	·6187
33	·6262	83	·6253	133	·6414	183	·6206
34	·6221	84	·6227	134	·6378	184	·6180
35	·6248	85	·6208	135	·6350	185	·6155
36	·6245	86	·6243	136	·6373	186	·6160
37	·6255	87	·6223	137	·6371	187	·6214
38	·6220	88	·6213	138	·6330	188	·6109
39	·6257	89	·6244	139	·6323	189	·6134
40	·6280	90	·6229	140	·6370	190	·6176
41	·6219	91	·6216	141	·6377	191	·6180
42	·6213	92	·6211	142	·6340	192	·6198
43	·6171	93	·6228	143	·6362	193	·6158
44	·6200	94	·6246	144	·6328	194	·6172
45	·6215	95	·6241	145	·6317	195	·6167
46	·6244	96	·6223	146	·6338	196	·6230
47	·6233	97	·6200	147	·6305	197	·6247
48	·6244	98	·6183	148	·6330	198	·6141
49	·6238	99	·6147	149	·6335	199	·6133
50	·6211	100	·6144	150	·6312	200	·6158
Mean diameter of coil A, 300·6209 mm.				Mean diameter of coil B, 300·6276 mm.			
Mean diameter of coils A and B, 300·6242 mm.							

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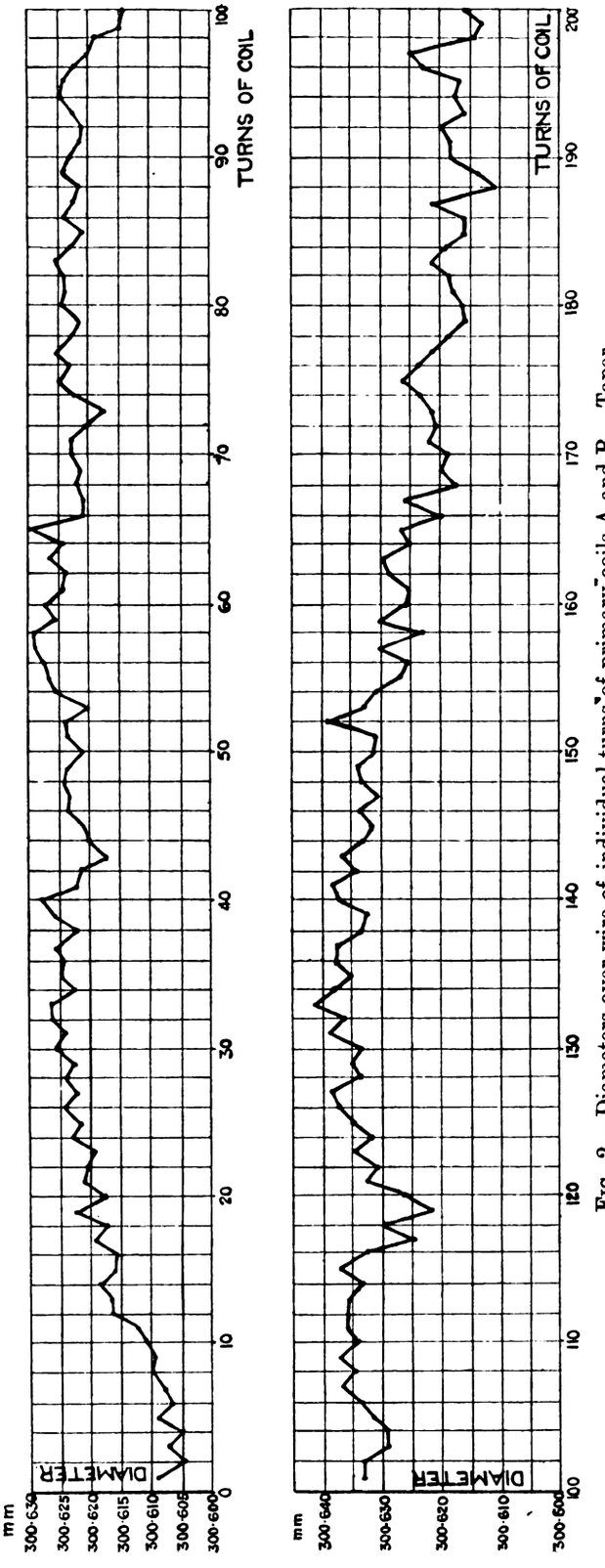

FIG. 2.—Diameters over wire of individual turns of primary coils A and B.—Taper.
(NOTE.—Mean diameter between axes is 0·6194 mm. smaller.)

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TABLE III.—Mean Diameters of Coils along Four Generators—Roundness.

Generator.	Mean diameter at 18° C.	
	Coil A.	Coil B.
	mm.	mm.
1	300·6244	300·6274
3	·6227	·6294
5	·6186	·6291
7	·6179	·6244
Means	300·6209	300·6276
Mean of coils A and B	300·6242 mm.	

A and B measured parallel to the axis of the cylinder, is 300 mm. The nominal distance of any point on any one turn from the corresponding point on the first turn of coil A is therefore readily determinable, and can be compared with the corresponding measured distance, the difference between the two—*i.e.*, measured distance minus nominal distance—giving the error of displacement with its appropriate sign.

The errors determined in this manner have been summarised in two ways. First, the mean value of the errors given by the four measurements made on each turn of the coils has been found, thus determining the displacement error of each turn. These mean errors are set out in Table IV, and are also shown graphically in Fig. 3. Secondly, the mean value of the errors as measured along each of the four generators has been found, thus determining how the mean error of displacement varies round each coil. These mean errors are set out in Table V.

These tables also show at the foot of each the mean error of displacement for each coil, and if both coils commenced and finished on the same generator, the displacement error of one with respect to the other would be found by subtracting the mean error for coil A from that for coil B. In the present case, however, owing to the fact that each coil commences and finishes on a different generator, a correction must be applied to this difference in accordance with the relative positions of these two generators. Let the angular separation of the generators be α, then the correction to be applied is equal to $\alpha p/2\pi$, where p is the pitch of the turns, the appropriate sign being applied. This correction amounts to $-0{\cdot}021$ mm.

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TABLE IV.—Mean Displacement Errors of each Turn of Coils A and B.

Coil A.				Coil B.			
Turn of coil.	Error at 18° C.	Turn of coil.	Error at 18° C.	Turn of coil.	Error at 18° C.	Turn of coil.	Error at 18° C.
	μ		μ		μ		μ
1	0	51	+ 6	101	+64	151	+65
2	− 1	52	+ 3	102	+68	152	+67
3	− 5	53	+ 4	103	+62	153	+74
4	− 3	54	− 1	104	+50	154	+69
5	−10	55	− 4	105	+51	155	+73
6	− 5	56	− 6	106	+50	156	+67
7	− 2	57	− 4	107	+48	157	+63
8	− 4	58	− 4	108	+49	158	+56
9	− 3	59	− 8	109	+50	159	+53
10	− 2	60	−13	110	+49	160	+48
11	− 2	61	−14	111	+59	161	+59
12	− 1	62	− 5	112	+63	162	+58
13	− 2	63	−11	113	+66	163	+62
14	− 7	64	−10	114	+65	164	+65
15	− 8	65	− 6	115	+63	165	+70
16	−11	66	−11	116	+61	166	+67
17	−13	67	−11	117	+65	167	+70
18	−15	68	−18	118	+63	168	+67
19	− 8	69	−18	119	+62	169	+64
20	− 7	70	−17	120	+51	170	+55
21	− 9	71	−20	121	+50	171	+58
22	− 9	72	−19	122	+54	172	+55
23	− 9	73	−14	123	+57	173	+59
24	− 5	74	−16	124	+58	174	+50
25	− 8	75	−15	125	+65	175	+56
26	− 6	76	−15	126	+65	176	+68
27	− 5	77	−13	127	+68	177	+73
28	− 4	78	− 7	128	+76	178	+72
29	− 6	79	− 8	129	+65	179	+66
30	− 9	80	− 7	130	+65	180	+66
31	− 8	81	−11	131	+57	181	+70
32	− 6	82	− 8	132	+56	182	+65
33	− 9	83	−12	133	+50	183	+63
34	− 7	84	−13	134	+53	184	+58
35	− 2	85	− 8	135	+57	185	+50
36	− 1	86	−14	136	+53	186	+46
37	− 2	87	− 8	137	+63	187	+50
38	− 1	88	− 7	138	+62	188	+54
39	+ 1	89	− 7	139	+74	189	+61
40	− 4	90	− 3	140	+67	190	+70
41	− 7	91	− 6	141	+67	191	+73
42	−12	92	− 8	142	+63	192	+66
43	− 8	93	− 2	143	+61	193	+66
44	− 9	94	− 5	144	+57	194	+65
45	− 8	95	−14	145	+54	195	+56
46	−11	96	−17	146	+56	196	+56
47	− 3	97	− 6	147	+57	197	+52
48	− 1	98	−12	148	+56	198	+45
49	− 1	99	− 8	149	+53	199	+47
50	+ 2	100	− 9	150	+59	200	+47
Mean displacement error of coil A = −7 μ.				Mean displacement error of coil B = +60 μ.			

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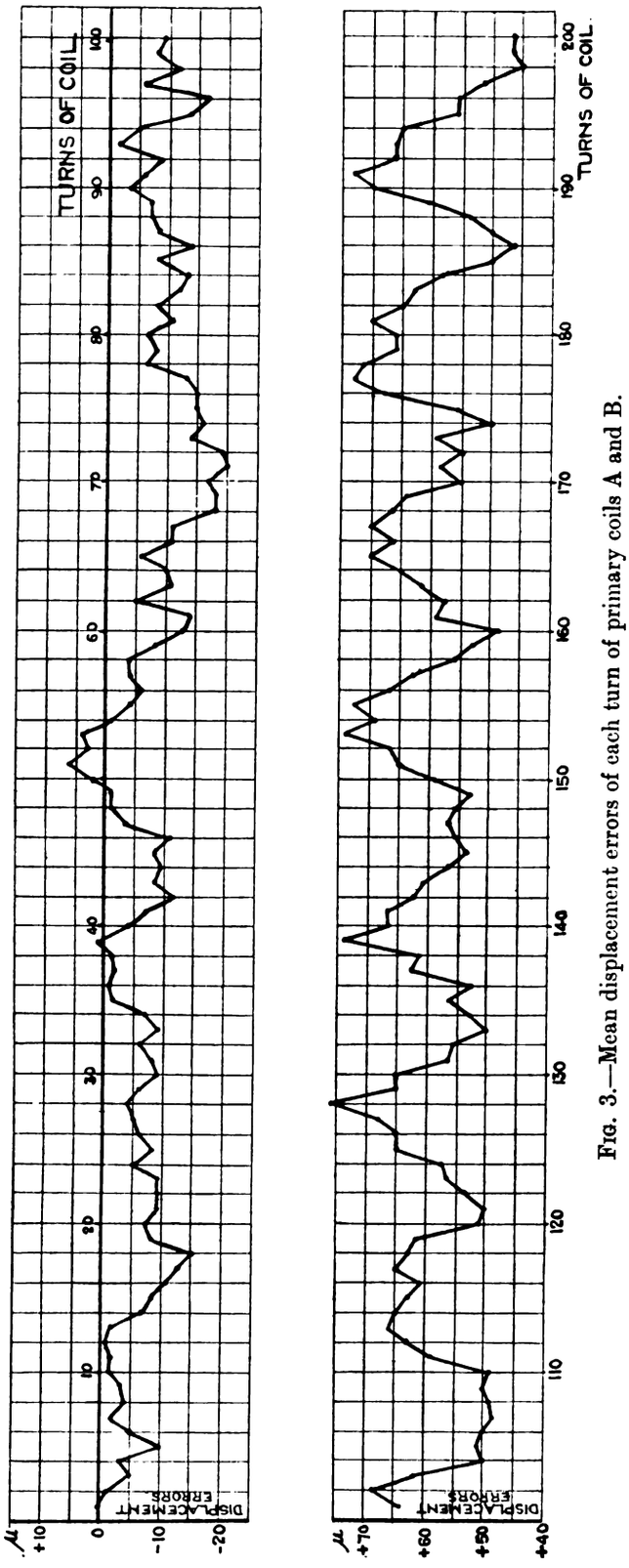

FIG. 3.—Mean displacement errors of each turn of primary coils A and B.

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TABLE V.—Mean Displacement Errors round each Primary Coil.

Generator.	Mean displacement error at 18° C.	
	Coil A.	Coil B.
	μ	μ
3	− 5	+64
7	−11	+58
11	− 3	+65
15	−10	+54
Means	− 7 μ	+60 μ

From these tables, together with the correction for position of leads, the mean distance between the two coils is—

300 mm. + 0·060 mm. — (—0·007) mm. — 0·021 mm., or 300·046 mm. at 18° C.

The actual mean pitch errors of the coils, and hence the mean distance separating adjacent turns, may be deduced from Tables IV and V by subtracting each displacement error from the one succeeding it.

Note on the First Turn of Coil A.—As noted on page 7 all the displacement errors have been expressed relative to the first turn of coil A, which for this purpose has been assumed to be a perfect helix. Though no measurements have been made to verify this, some idea of the extent of the departure from a true helix may be inferred from Table V. This table may be recast by expressing the errors along each generator in terms of that of generator 3, the relative errors thus obtained being shown in Table VI.

TABLE VI.

Generator.	Relative displacement error at 18° C.		
	Coil A.	Coil B.	Mean.
	μ	μ	μ
3	0	0	0
7	− 6	− 6	− 6
11	+ 2	+ 1	+ 1·5
15	− 5	−10	− 7·5

These figures cannot be taken as a definite measure of the errors of the first turn since they are only relative to the mean of all the other turns (which may or may not itself correspond to a true helix), and, further, are affected by the observational errors

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inevitably associated with the readings on a single turn of the coil. They serve, however, to give an indication of the general degree of accuracy to be associated with any individual turn.

Summary.

(i) *Coefficient of Expansion of Cylinder.*

(*a*) Axial	0·0000065 per 1° C.
(*b*) Diametral	0·0000035 per 1° C.

(ii) *Mean Diameter of Wire.*

For each coil	0·6194 mm.

(iii) *Mean Diameters of Coils at* 18° *C.*

	Over coils.	*Between axes.*
Coil A	300·6209 mm.	300·0015 mm.
Coil B	300·6276 mm.	300·0082 mm.
Coils A and B	300·6242 mm.	300·0048 mm.

(iv) *Axial Spacing of Coils at* 18° *C.*

Mean distance between coils	300·046 mm.

IV. Calculations of the Value of the Standard.

The above measurements all refer to the axis of the wire. Irregularities both of diameter and of axial spacing of the turns were noted, but as a first approximation the coils were assumed to be uniform in diameter and axial spacing. Thus, in the first place, the calculations were made for an ideal standard having the following dimensions :—

Diameter of each primary coil	= 30·00048 cm.
Distance between inside turns	= 20·0052 cm.
Length of each coil	= 10·0000 cm.
Number of turns in each coil	= 100.

Diameter of the Neutral Circle.—The finding of this diameter is the first step in the calculation.

The formulæ used were those due to J. Viriamu Jones,* giving the mutual inductance between a helix and a circle in the plane of one end. If we have a helix of radius a and axial length x, then the mutual inductance M_x between it and a circle of radius A in the plane of one end is given by—

$$M_x = \theta (A + a) ck \left[\frac{K - E}{k^2} + \left(\frac{c'}{c} \right)^2 (K - \Pi) \right], \quad \ldots \ldots \quad (1)$$

* J. V. Jones, ' Proc. Roy. Soc.', A, Vol. 63, p. 198, 1898.

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where θ is the angular length of the helix (*i.e.*, $\theta = 2\pi n_1$, where n_1 is the number of turns),

$$c^2 = \frac{4Aa}{(A+a)^2}, \qquad c'^2 = 1 - c^2, \qquad k^2 = \frac{4Aa}{(A+a)^2 + x^2},$$

K, E and Π are complete elliptic integrals of the first, second and third kinds respectively; K and E are to modulus k, and

$$\Pi = \int_0^{\pi/2} \frac{d\phi}{(1 - c^2 \sin^2 \phi)(1 - k^2 \sin^2 \phi)^{1/2}}.$$

We have also

$$\frac{\partial M_x}{\partial A} = \theta ck\left[K + \frac{A-a}{2a}(K - \Pi)\right]. \quad \ldots\ldots\ldots\ldots \quad (2)$$

For the purposes of calculation each of the primary coils was regarded as the difference of two helices, of the same diameter and pitch and of lengths x_1 and x_2, where x_1 and x_2 are the axial distances of the two ends of the coil from the plane of the secondary circle or coil.

If M_{x_1} and M_{x_2} are the values of the mutual inductance between the secondary circle and the two imaginary helices, then the mutual inductance between the actual coil and the secondary coil is given by—

$$M = M_{x_2} - M_{x_1}.$$

Similarly

$$\frac{\partial M}{\partial A} = \left(\frac{\partial M_x}{\partial A}\right)_{x_2} - \left(\frac{\partial M_x}{\partial A}\right)_{x_1}.$$

The values of M_{x_2}, M_{x_1}, $(\partial M_x/\partial A)_{x_1}$ and $(\partial M_x/\partial A)_{x_2}$ are given directly by equations (1) and (2). It should be noticed that if p is the pitch of the actual coil winding, the number of turns in the two imaginary helices must be taken as x_1/p and x_2/p respectively. Having calculated the values of M and $\partial M/\partial A$ for one primary coil in this way, the values must then be doubled in order to obtain the values of M and $\partial M/\partial A$ for the complete primary winding.

The actual forms of equations (1) and (2) used in the calculations were as follows:—

$$M_x = 2\pi^2 n_1 n_2 (A+a)\left[\frac{c}{k}\left(\frac{K-E}{\pi}\right) + \frac{(A-a)^2}{A+a} \cdot \frac{1}{r_1}\left(\frac{K-\Pi}{\pi}\right)\right] \quad \ldots\ldots \quad (3)$$

$$\left(\frac{\partial M_x}{\partial A}\right) = 2\pi^2 n_1 n_2 ck\left[\frac{K}{\pi} + \frac{A-a}{2a}\left(\frac{K-\Pi}{\pi}\right)\right]. \quad \ldots\ldots\ldots \quad (4)$$

Here n_2 is the number of secondary turns, which as a first approximation are considered to be concentrated in the circumference of the neutral circle. The number of primary turns n_1 in the particular imaginary helix under consideration is given by x/p, where x is the axial length and p is the pitch of the winding.

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The elliptic integrals K, E and Π required were calculated by the methods of L. V. KING,* which make use of the scale of arithmetic-geometrical means, which is as follows, starting with positive numbers,

$$\begin{array}{lll} a_1 = \tfrac{1}{2}(a_0 + b_0), & b_1 = \sqrt{a_0 b_0}, & c_1 = \tfrac{1}{2}(a_0 - b_0), \\ a_2 = \tfrac{1}{2}(a_1 + b_1), & b_2 = \sqrt{a_1 b_1}, & c_2 = \tfrac{1}{2}(a_1 - b_1), \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}), & b_n = \sqrt{a_{n-1} b_{n-1}}, & c_n = \tfrac{1}{2}(a_{n-1} - b_{n-1}). \end{array}$$

The a's and b's tend to the same limit, which for convenience may be denoted by a_n. The calculations were carried out to eight significant figures, and with this accuracy it was found that in all cases $a_n = a_4 = b_4$. The integrals K and E are obtained as follows: The A.G.M. scale $a_0 = 1$, $b_0 = k' = \sqrt{1 - k^2}$, $c_0 = k$ was constructed, and the limit a_n found. Then we have

$$\frac{K}{\pi} = \frac{1}{2a_n} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

$$\frac{K - E}{\pi} = \frac{1}{4a_n}[c_0^2 + 2c_1^2 + 4c_2^2 + \ldots]. \quad \cdots\cdots\cdots \quad (6)$$

The calculation of Π was performed thus :—

(*a*) The angle θ was calculated from the equation—

$$\sin^2\theta = \frac{x^2}{(A - a)^2 + x^2}.$$

(*b*) Starting with $\psi_0 = \theta$, the angles ψ_1, ψ_2, ψ_3, etc., were obtained using the relation

$$\sin(2\psi_{n+1} - \psi_n) = \frac{b_n}{a_n}\sin\psi_n.$$

The a's and b's are taken from the A.G.M. scale previously constructed.

(*c*) The summation $\sum_2(\psi_n c_n)$ was then computed, where

$$\sum_2(\psi_n c_n) = 2c_2\frac{\tan(2\psi_2 - \psi_1)}{\cos(2\psi_3 - \psi_2)} + 6c_3\frac{\tan(2\psi_3 - \psi_2)}{\cos(2\psi_4 - \psi_3)} + 6c_4\frac{\tan(2\psi_4 - \psi_3)}{\cos(2\psi_5 - \psi_4)} + \ldots + 2(2^n - 1)\,c_{n+2}\frac{\tan(2\psi_{n+2} - \psi_{n+1})}{\cos(2\psi_{n+3} - \psi_{n+2})}.$$

(*d*) Finally,

$$\Pi = \frac{2K\cos(2\psi_1 - \psi_0)}{k'^2\sin 2\psi_0}\left[a_n\sin\psi_n + \sum_2(\psi_n c_n)\right]. \quad \cdots\cdots\cdots \quad (7)$$

* L. V. KING, 'On the Direct Numerical Calculation of Elliptic Functions and Integrals', Cambridge University Press (1924).

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The series $\sum_2 (\psi_n c_n)$ was rapidly convergent. In working to eight figure accuracy only the first three terms were significant, and of these the first was required to seven significant figures, the second to five, and the third to one only.

The plane of the neutral circle is midway between the two primary coils, and is, therefore, 10·0026 cm. distant from the inside turn of each primary coil. The length of each coil being 10·000 cm., the far end of the coil will be at a distance of 20·0026 cm. away from the plane of the neutral circle. Thus the Jones formula must be applied to two imaginary helices of axial lengths 10·0026 cm. and 20·0026 cm. respectively. The pitch of the winding is 1 mm. in each case, so that the numbers of turns must be taken as 100·026 and 200·026 respectively. The radius of the neutral circle is the value of A which satisfies the relation $\partial M/\partial A = 0$, where M refers to the mutual inductance between the secondary coil and the complete primary coil. The value of A for the neutral circle was known from previous calculations to be approximately 25·7 cm. The actual value of $\partial M/\partial A$ was computed for the following values of A, viz., 25·709, 25·706 and 25·7034. The values are shown in Tables VII and IX.

The actual value of the radius of the neutral circle was derived from these values, making use of an equation obtained by WILMOTTE,* which gives the change in mutual inductance due to a small displacement of the secondary coil, or a small change in the diameter of the secondary coil. Let A_0 be the radius of the neutral circle and M_0 the value of the mutual inductance when the secondary coil lies on the circumference of this circle. If now the radius of the secondary circle is increased by a small amount ξ, we have for the corresponding change of mutual inductance

$$M - M_0 = 2\pi n_2 A_0 \left[\frac{\partial R}{\partial x}\xi^2 + \frac{1}{3}\left\{\frac{\partial^2 R}{\partial x\,\partial A} + \frac{2}{A_0}\frac{\partial R}{\partial x}\right\}\xi^3 + \ldots\right], \quad \ldots \quad (8)$$

where R is the radial component of magnetic force at any point P on the secondary circle due to one of the primary coils carrying unit current. As before x represents the axial distance of the secondary coil from the primary coil. For the calculation of the coefficients of ξ^2 and ξ^3 we have the following relations :—

$$\frac{\partial R}{\partial x} = \left[n_3 R'\right]_{x_1}^{x_2}, \quad \ldots \quad (9)$$

$$\frac{\partial^2 R}{\partial x\,\partial A} = \left[n_3 \frac{\partial R'}{\partial A}\right]_{x_1}^{x_2}. \quad \ldots \quad (10)$$

R′ is the radial component of magnetic force at any point P on the secondary circle due to unit current in a single turn of the primary coil, this turn being at an axial distance x from the secondary circle; x_1 is the value of x for the turn nearest to the secondary

* R. M. WILMOTTE, *loc. cit.*

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TABLE VII.—$a = 15 \cdot 00024$ cm.

A.	$x_2 = 20 \cdot 0026$ cm.				$x_1 = 10 \cdot 0026$ cm.				$\frac{\partial M/\partial A}{4\pi^2 n_2 c}$.
	k^2.	$\frac{K}{\pi}$.	$\frac{K-\Pi}{\pi}$.	$\frac{(\partial M/\partial A)_{x_2}}{2\pi^2 n_2 c}$.	k^2.	$\frac{K}{\pi}$.	$\frac{K-\Pi}{\pi}$.	$\frac{(\partial M/\partial A)_{x_1}}{2\pi^2 n_2 c}$.	
cm.									
25·7060	0·74978522	0·68631781	−2·5282078	−37·39113	0·87782629	0·79076391	−3·3343288	−37·40247	0·01134
25·7090	0·74978372	0·68631693	−2·5274547	−37·38847	0·87780674	0·79074012	−3·3331115	−37·39480	0·00633
25·7034	0·74978655	0·68631855	−2·5288607	−37·39342	0·87784327	0·79078455	−3·3353854	−37·40913	0·01571

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circle, *i.e.*, the first turn of the primary coil, while x_2 is the last turn. R′ is given by a formula due to A. RUSSELL,* viz.:—

$$R' = \frac{2}{r_1}\left[\frac{2ax}{r_2^2}E - \frac{x}{A}(K - E)\right], \quad \ldots\ldots\ldots\ldots \quad (11)$$

where

$$r_1^2 = (A + a)^2 + x^2,$$
$$r_2^2 = (A - a)^2 + x^2.$$

K and E are complete elliptic integrals to modulus $k = \frac{\sqrt{4Aa}}{r_1^2}$, of the first and second kinds respectively. Values of $\partial R/\partial x$ were computed from equations (9) and (11), taking for A the values 25·509, 25·709 and 25·909. Sufficient accuracy was obtainable in these calculations by using for K and E the tables due to NAGAOKA and SAKURAI.† The values are given in Table VIII.

TABLE VIII.—$a = 15{\cdot}00024$ cm., $n_3 = 10$, $x_2 = 20{\cdot}0026$ cm., $x_1 = 10{\cdot}0026$ cm.

	A = 25·509 cm.	A = 25·909 cm.	A = 25·709 cm.
R'_{x_2}	0·0302110	0·0294667	0·0298384
R'_{x_1}	0·0509268	0·0482323	0·0495609
$\partial R/\partial x$	−0·207158	−0·187656	−0·197225

From these results we deduce that for A = 25·71 cm. (the radius of the neutral circle) we have

$$\frac{\partial R}{\partial x} = -0{\cdot}197_2.$$

$$\frac{\partial^2 R}{\partial x\,\partial A} = 0{\cdot}048_7,$$

$$\frac{1}{3}\left\{\frac{\partial^2 R}{\partial x\,\partial A} + \frac{2}{A_0}\frac{\partial R}{\partial x}\right\} = 0{\cdot}011_1.$$

Thus equation (8) becomes

$$M - M_0 = 2\pi A_0 n_2 [-0{\cdot}197_2\,\xi^2 + 0{\cdot}011_1\,\xi^3 + \ldots].$$

The number of secondary turns n_2 was provisionally fixed as 462, *i.e.*, 22 complete layers of 21 turns each. Substituting this value we get

$$M - M_0 = [-14{\cdot}71_5\,\xi^2 + 0{\cdot}8_3\,\xi^3 + \ldots]10^3, \quad \ldots\ldots\ldots \quad (12)$$

* A. RUSSELL, 'Proc. Phys. Soc. Lond.', Vol. 20, p. 478, 1907.

† NAGAOKA and SAKURAI, 'Sci. Papers, Inst. Phys. Chem. Research, Tokyo', Vol. 2, p. 1, 1922.

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whence by differentiation, remembering that $\partial M/\partial A = \partial M/\partial \xi$,

$$\frac{\partial M}{\partial A} = [-29{\cdot}4_3\,\xi + 2{\cdot}4_9\,\xi^2 + \ldots]10^3.*$$

The values of $\partial M/\partial A$ derived from Table VII were substituted in this equation and the corresponding values of ξ determined. Evidently the radius of the neutral circle is given by $A_0 = A - \xi$. The values obtained are given in Table IX.

TABLE IX.

A.	$\partial M/\partial A$.	ξ.	$A_0 = A - \xi$.
cm.		cm.	cm.
25·7090	$111{\cdot}_4$	$-0{\cdot}0037_8$	$25{\cdot}7127_8$
25·7060	$199{\cdot}_8$	$-0{\cdot}0067_8$	$25{\cdot}7127_8$
25·7034	$276{\cdot}_4$	$-0{\cdot}0093_0$	$25{\cdot}7127_0$

Thus the radius of the neutral circle is 25·7128 cm.

Evaluation of the Jones Formula for M_0.—Equation (3) allows the mutual inductance M to be calculated for any desired value of the secondary radius A. When A is nearly equal to A_0, the value of M_0 may be obtained from the value of M by means of equation (12). Two independent calculations of M_0 were made in this way. Values of M were calculated for A = 25·709 and 25·706, using equation (3), and the corresponding values of $M - M_0$ calculated by (12). The results are given in Tables X and XI.

TABLE XI.

A.	$\frac{M}{4\pi^2 n_2 (A+a)}$.	M.	ξ.	$M - M_0$.	M_0.
cm.		cm.	cm.	cm.	cm.
25·706	13·395447	$9945346{\cdot}_5$	−0·00678	−1·35	$9945347{\cdot}_8$
25·709	13·394461	$9945347{\cdot}_2$	−0·00378	−0·42	$9945347{\cdot}_6$

Thus, if there were 462 secondary turns and these were concentrated round the circumference of the neutral circle, the value of the mutual inductance would be 9,945,348 cm.

The calculated value M = 9,945,348 cm. was considered not quite near enough to 10^7 cm. for convenience in use, and accordingly it was decided to add an additional three secondary turns in order to make the value a little greater than 10 millihenries or 10,000,000 cm. These three turns were wound so that one touched the side of the groove containing the

* The equation for $\partial M/\partial A$ of the original model of the Campbell mutual inductance standard was $\partial M/\partial A = -52{\cdot}_2\,\xi + 7\xi^2$. Thus the errors associated with the dimensions of the secondary coil in the present model are likely to be about half those in the original.

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TABLE X.—$a = 15 \cdot 00024$ cm.

A.	$x_2 = 20 \cdot 0026$ cm.				$x_1 = 10 \cdot 0026$ cm.				$\frac{M}{4\pi^2 n_2 (A+a)}$.
	k^2.	$\frac{K-E}{\pi}$.	$\frac{K-\Pi}{\pi}$.	$\frac{Mx_2}{2\pi^2 n_2 (A+a)}$.	k^2.	$\frac{K-E}{\pi}$.	$\frac{K-\Pi}{\pi}$.	$\frac{Mx_1}{2\pi^2 n_2 (A+a)}$.	
cm.									
25·706	0·74978522	0·30078362	−2·5282078	35·642013	0·87782629	0·43348275	−3·3343288	22·246566	13·395447
25·709	0·74978372	0·30078245	−2·5274547	35·636730	0·87780674	0·43345417	−3·3331115	22·242269	13·394461

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secondary coil, and the others were as near to it as possible, the three being in the same layer. The additional mutual inductance due to these three turns was calculated by using WILMOTTE's formula (*cf.* equation 8) and the value of M_0. WILMOTTE's complete formula is

$$M' - M'_0 = 2\pi A_0 \left[\frac{\partial R}{\partial x}(\xi^2 - \eta^2) + \tfrac{1}{3}\left(\frac{2}{A_0}\frac{\partial R}{\partial x} + \frac{\partial^2 R}{\partial x\,\partial A}\right)\xi^3 - \left(\frac{1}{A_0}\frac{\partial R}{\partial x} + \frac{\partial^2 R}{\partial x\,\partial A}\right)\xi\eta^2 + \ldots \right], \quad . . \quad (13)$$

where M' is the mutual inductance between the primary coils and a secondary circle of radus $A_0 + \xi$, the plane of this circle being at a distance η from that of the neutral circle. M'_0 is the mutual inductance between the primary coils and the neutral circle. The other quantities are the same as in equation (8), and their values have already been obtained. Inserting the numerical values we have

$$M' - M'_0 = 2\pi A_0 [0{\cdot}197_2\,(\xi^2 - \eta^2) + 0{\cdot}011_1\,\xi^3 - 0{\cdot}041\,\xi\eta^2 + \ldots].$$

The three extra turns may with sufficient accuracy be replaced by an equivalent mean circle, round the circumference of which the turns may be considered to be concentrated. For this equivalent circle we have $\xi = 0{\cdot}530$ cm., $\eta = 0{\cdot}438$ cm. Substituting in the above equation we find that for the three turns $M - M_0 = -9{\cdot}7$. Now M_0 for three turns is $(3/462) \times 9{,}945{,}348 = 64580{\cdot}2$ cm. Thus the actual mutual inductance due to the three turns is $64{,}570{\cdot}_5$ cm., and the total value of M_0 for the 465 turns is

$$M_0 = 10{,}009{,}918 \text{ cm.}$$

The calculation of this value assumes that 462 of the turns are concentrated in the circumference of the neutral circle, and also neglects irregularities in the diameter and pitch of the primary windings. Corrections for these irregularities and for the finite area of the cross section of the secondary winding must now be applied.

Corrections for Variations of Diameter and Pitch of Primary Windings.—For the purpose of making corrections for irregularities in the primary winding, each turn of this winding was regarded as a complete circle of radius a, at a distance x from the plane of the secondary coil, which was regarded as another circle of radius A. For two such circles we have the following formulæ* :—

$$\frac{\partial M}{\partial a} = 2\pi a X = \frac{4\pi a}{r_1}\left\{\frac{2A\,(A-a)}{r_2^2}E + (K - E)\right\}, \quad . . . . . \quad (14)$$

$$\frac{\partial M}{\partial x} = -2\pi a R = \frac{4\pi a}{r_1}\left\{\frac{2Ax}{r_2^2}E - \frac{x}{a}(K - E)\right\}. \quad . . . . . \quad (15)$$

Here M denotes the mutual inductance of the two circles; X and R are the axial and radial components, respectively, of the magnetic force at P, any point on the primary circle of radius a, due to unit current flowing round the secondary circle of radius A.

* A. RUSSELL, 'Proc. Phys. Soc. Lond.', Vol. 20, p. 478, 1907.

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K and E are complete elliptic integrals of the first and second kinds respectively to modulus $k = \sqrt{4Aa/r_1^2}$. Also $r_1^2 = (A + a)^2 + x^2$ and $r_2^2 = (A - a)^2 + x^2$. If, now, the mean radius of any turn of the primary coil exceeds the mean value for all the coils (15·00024 cm.) by an amount Δa, then the mutual inductance (for one secondary turn) will be increased by an amount which is given approximately by $\frac{\partial M}{\partial a} . \Delta a$, where $\partial M/\partial a$ is given by equation (14). Similarly, if the axial distance of any turn from the secondary coil differs from its nominal value by an amount Δx, there will be a corresponding increment of mutual inductance given by $\frac{\partial M}{\partial x} . \Delta x$, $\partial M/\partial x$ being given by (15). $\partial M/\partial a$ and $\partial M/\partial x$ were calculated for a few turns, and curves plotted showing the relation between these quantities and x. From these curves the values of $\partial M/\partial a$ and $\partial M/\partial x$ could be obtained for any individual turn. Since the complete primary winding is symmetrical about the plane of the secondary coil, corresponding turns on the two primary coils were dealt with together. Their corrections (Δa or Δx) were added together and a net correction found. The curves of $\partial M/\partial a$ and $\partial M/\partial x$ are shown in Figs. 4 and 5. The values of the diameters of the various turns of the primary coils are given in Table II. The corresponding values of Δa were obtained from these. The values of Δx were deduced from Table IV, assuming the corrections on turns 100 and 101 to be zero, since in the main calculations the value of x assumed for these turns was the actual measured value. Thus for coil A the value of Δx for each turn was referred to turn 100 as basis, while for coil B turn 101 formed the basis. Multiplying corresponding values of $\partial M/\partial a$ and Δa, the value of the correction ΔM for each turn was found, and the resultant correction for the whole primary winding found by addition. Similarly the correction for axial displacements was obtained. The corrections referring to one secondary turn were then multiplied by the total number of secondary turns, viz., 465, in order to get the total correction. The final results were :—

Total correction due to variations of diameter of primary turns $= + 56{\cdot}_8$ cm.

Total correction due to axial displacements of primary turns $= + 159$ cm.

Correction for Finite Cross Section of Secondary Winding.—This correction is most simply evaluated by using a formula due to SEARLE, viz. :—

$$M = M_0 + \frac{1}{A}\frac{\partial M}{\partial A} + \left(\frac{B^2 - C^2}{6} + \frac{9C^4 - 20B^2C^2}{360A^2}\right)\frac{\partial^2 M}{\partial A^2} + \ldots, \quad . \quad . \quad . \quad . \quad (16)$$

where M_0 is the value of the mutual inductance when all the secondary turns are concentrated in the mean circle of the secondary winding, which is of radius A. M is the value of the mutual inductance when the secondary turns are uniformly distributed over a rectangular cross-section of winding, of radial depth 2B and of axial length 2C.

The value of $\partial M/\partial A$ is known to be zero for the mean circle of the secondary winding

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(the neutral circle). The value of $\partial^2 M/\partial A^2$ was obtained by differentiating equation (12) twice. We have $\partial^2 M/\partial A^2 = -29{\cdot}4_3 \times 10^3$. Also by actual measurements, allowing for the thickness of the silk insulation, we have B = 0·521 cm., C = 0·508 cm. Substituting in (16) the actual correction was found to be — 66 cm.

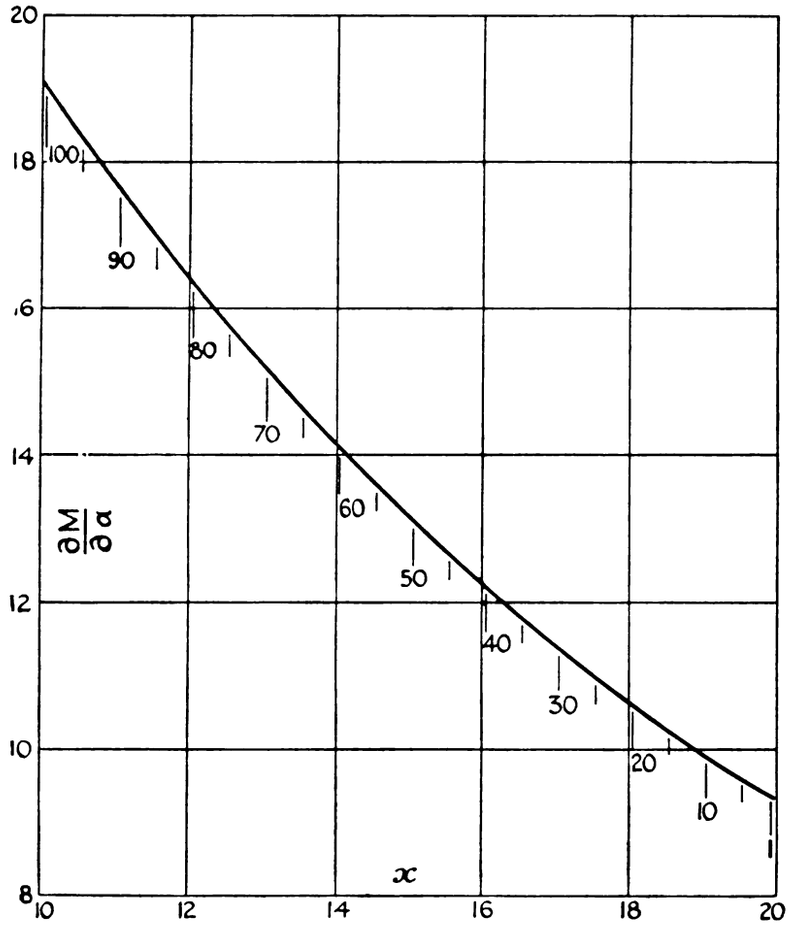

FIG. 4.—Relation between $\partial M/\partial a$ for any single turn of the primary winding and its axial distance from the mean plane of the secondary coil.

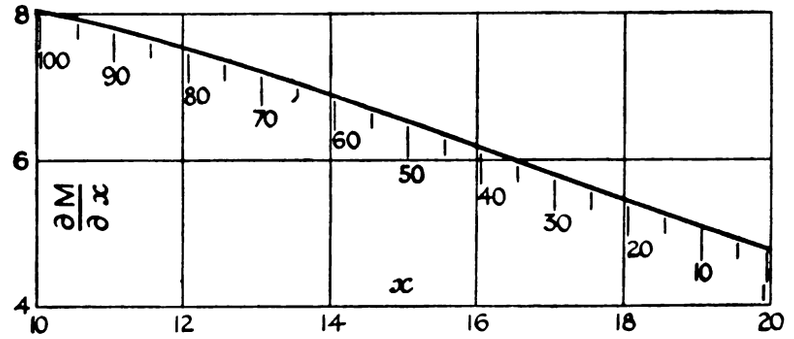

FIG. 5.—Relation between $\partial M/\partial x$ for any single turn of the primary winding and its axial distance from the mean plane of the secondary coil.

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This value was checked by another calculation, in which the secondary winding was replaced by nine equivalent circles distributed over the cross section of the winding in a manner suggested by S. BUTTERWORTH,* an appropriate fraction of the total turns being associated with each circle. The value of the correction for each circle was obtained from equation (13), and then the total correction obtained by addition. The distribution of the circles and turns is given in the following Table XII :—

TABLE XII.

Circle No.	ξ.	η.	Associated turns.
1	0	0	64N/324
2	0	$-C\sqrt{0\cdot6}$	40N/324
3	0	$+C\sqrt{0\cdot6}$	40N/324
4	$-B\sqrt{0\cdot6}$	0	40N/324
5	$+B\sqrt{0\cdot6}$	0	40N/324
6	$-B\sqrt{0\cdot6}$	$-C\sqrt{0\cdot6}$	25N/324
7	$-B\sqrt{0\cdot6}$	$+C\sqrt{0\cdot6}$	25N/324
8	$+B\sqrt{0\cdot6}$	$-C\sqrt{0\cdot6}$	25N/324
9	$+B\sqrt{0\cdot6}$	$+C\sqrt{0\cdot6}$	25N/324

N (= 462) = total number of turns in rectangular cross-section of winding.

The value of the correction obtained by this method was — 65 cm.

We have now for the final value of the mutual inductance :—

M_0 for 462 turns of secondary winding	=	9,945,348 cm.
M for three additional turns of secondary winding ..	=	64,570 cm.
Radial corrections for primary winding	=	57 cm.
Axial corrections for primary winding	=	159 cm.
		10,010,134 cm.
Deduct correction for finite cross-section of secondary winding	=	65 cm.
Hence final value of the standard of mutual inductance ..	=	10,010,069 cm.

THE MUTUAL INDUCTANCE OF THE STANDARD = 10,010·07 MICROHENRIES AT 18° C.

Temperature Coefficient of Mutual Inductance.—The expansion of the primary winding with rise of temperature may be assumed to follow that of the marble, since the wire is wound on the cylinder under considerable tension. The values of the coefficient of

* S. BUTTERWORTH, 'Glazebrook's Dictionary of Applied Physics', Vol. 2, p. 387. Macmillan, London (1922).

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expansion of the marble are $6{\cdot}5 \times 10^{-6}$ per 1° C. for the axial direction, and $3{\cdot}5 \times 10^{-6}$ per 1° C. for the radial direction. Thus, if θ denote temperature, we have—

$$\frac{1}{a}\cdot\frac{da}{d\theta} = 3{\cdot}5 \times 10^{-6} \quad \text{and} \quad \frac{1}{x}\frac{dx}{d\theta} = 6{\cdot}5 \times 10^{-6}.$$

Also $a = 15$, and for x we may take a mean value of 14. Thus $da/d\theta = 5{\cdot}3 \times 10^{-5}$ and $dx/d\theta = 9{\cdot}0 \times 10^{-5}$. From the curves of Figs. 4 and 5 we find that as mean values of $\partial M/\partial a$ and $\partial M/\partial x$ for one primary turn and one secondary turn we may take the values 14 and $-6{\cdot}5$ respectively. In order to get the actual equivalent values of $\partial M/\partial a$ and $\partial M/\partial x$, we have to multiply these by the product of the total numbers of primary and secondary turns (200×465). Thus we find

$$\frac{\partial M}{\partial a} = 13 \times 10^5 \quad \text{and} \quad \frac{\partial M}{\partial x} = -6 \times 10^5.$$

For the change of mutual inductance with temperature we have

$$\frac{dM}{d\theta} = \frac{\partial M}{\partial a}\cdot\frac{da}{d\theta} + \frac{\partial M}{\partial x}\cdot\frac{dx}{d\theta} + \frac{\partial M}{\partial A}\cdot\frac{dA}{d\theta}.$$

The value of $\partial M/\partial A$ is known to be zero.

Substituting the values given above for the other quantities we have

$$\frac{dM}{d\theta} = 13 \times 10^{-5} \times 5{\cdot}3 \times 10^5 - 6 \times 10^5 \times 9 \times 10^{-5}$$
$$= 15 \text{ cm. per } 1^\circ \text{ C.}$$

The temperature coefficient of mutual inductance $\left(\frac{1}{M}\cdot\frac{dM}{d\theta}\right)$ is therefore $1{\cdot}_5 \times 10^{-6}$. It will be observed that the difference in the expansion of the marble in the two directions has a compensating effect on the changes of mutual inductance.

Probable Errors.—It is difficult to estimate with certainty the probable errors that are likely to enter into the measurements and calculations, but the following considerations will give a rough guide as to the overall accuracy which may be expected :—

(1) Primary diameter.—The mean error on the measurements of the primary diameter is probably $\pm\ 0{\cdot}0001$ cm. Thus the mean error on the radius Δa may be taken as 5×10^{-5} cm. We have seen that $\partial M/\partial a$ may be assumed to be 13×10^{-5}. Thus the corresponding error, ΔM, on the mutual inductance is 65 cm. or $6{\cdot}5$ parts in a million.

(2) Axial distances.—The dimension actually measured was the distance between corresponding turns on the two primary coils. This is roughly $2x$. The probable error on the measurement was $\pm\ 0{\cdot}0002$ cm., giving $\Delta x = \pm\ 0{\cdot}0001$. The

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value of $\partial M/\partial x$ has been shown to be -6×10^5. Hence the corresponding error, ΔM, is $\pm$ 60 cm., or six parts in a million.

(3) Secondary diameter.—This was probably correct to 0·1 mm. Hence $\Delta A = \pm 0{\cdot}005$ cm. The corresponding error, ΔM, is given by equation (12) as $\pm 0{\cdot}4$ cm., or 0·04 part in a million.

(4) Width and depth of secondary winding (2B and 2C).—These were probably correct to $\pm 0{\cdot}005$ cm., giving $\Delta B = \Delta C = \pm 2{\cdot}5 \times 10^{-3}$ cm. From equation (16) we have $\frac{\partial M}{\partial B} = \frac{B}{3}\frac{\partial^2 M}{\partial A^2}$ (approx.), the other term being negligible. Inserting the numerical values we find $\partial M/\partial B = 5 \times 10^3$. The corresponding error, ΔM, is thus $\pm$ 13 cm. or 1·3 parts in a million. The error associated with C is the same as this. Thus the total error associated with the cross-section of the secondary coil is $\pm 2{\cdot}_6$ parts in a million.

The probable errors are collected together for convenience in the following table :—

TABLE XIII.—Table of Probable Errors.

Dimension.	Probable error. Parts in a million.
Primary diameter	$\pm 6{\cdot}_5$
Axial length	$\pm 6{\cdot}_0$
Secondary diameter	$\pm 0{\cdot}0_4$
Secondary cross-section	$\pm 2{\cdot}_6$

Thus the maximum probable error is about 15 parts in a million.

V. EXPERIMENTAL COMPARISON WITH NATIONAL PHYSICAL LABORATORY STANDARD.

The method used for the comparison consists in placing the two standards in opposition with the same current flowing through their primary coils. The difference between their values is balanced out by means of a very low-range variable mutual inductance. Fig. 6 is a diagrammatic representation of the circuit. The two standards are denoted by 1 and 2, M_1 and M_2 being the values of their mutual inductances. The small variable mutual inductance is denoted by 3, M_3 representing its value. The phase angles of the two standards were not exactly equal, so that in order to obtain a perfect balance, as indicated by the vibration galvanometer G, it was necessary to use a very small potentiometer slide-wire resistance, R in Fig. 6. This normally had a total resistance of about 0·01 ohm, but its terminals were joined by a number of strands of copper wire, so that

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the resistance actually included in the circuit was probably of the order of 0·00001 ohm. The actual value of the resistance was not required. Current of very low frequency (10 cycles per second) was used, so that capacity effects were extremely small, and the effective mutual inductance was identical with the geometrical value within the limits of practical measurement. The detector G was a Campbell vibration galvanometer, tuned to a frequency of 10 cycles per second. This was found to be very susceptible

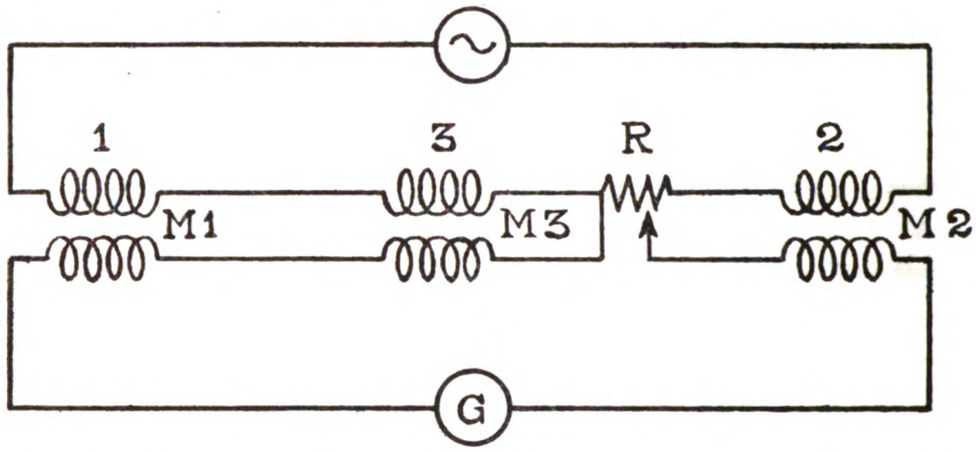

Fig. 6.—Circuit used for experimental comparison of the two primary standards of mutual inductance.

to vibrations of the building, but by placing the galvanometer on a slate slab suspended by four lengths of rubber tubing these undesirable vibrations were almost completely eliminated and the working sensitivity was greatly increased. The primary current used was 0·2 ampere, and with this current, using the phase-adjusting resistance and anti-vibration support mentioned above, a sensitivity of about one part in a million could be obtained. This could be read on the low range Campbell inductometer M_3 (Fig. 6).

In working to this accuracy the various cross mutual inductances must be taken into account. For example, in addition to the three mutual inductances, M_1, M_2, M_3, shown in Fig. 6, there will also be a mutual inductance, m_{13} say, between the primary of 1 and the secondary of 3, a mutual inductance m_{31} between the primary of 3 and the secondary of 1, and so on. There will evidently be 9 mutual inductances in all, the three main ones, M_1, M_2, M_3, and the six cross inductances m_{13}, m_{12}, m_{31}, m_{32} m_{21}, m_{23}. Thus, instead of the equation of balance taking the simple form $M_1 - M_2 = M_3$, we find that it becomes

$$M_1 - M_2 = M_3 + m_{12} + m_{13} + m_{21} + m_{23} + m_{31} + m_{32}.$$

The signs to be attached to m_{12}, m_{13}, etc., are unknown, but they are all written as positive for convenience. It will be seen later that their actual signs are immaterial. Having balanced the bridge and noted the reading of M_3, the primary and secondary coils of 1 are reversed and balance again obtained. The equation now becomes

$$M_1 - M_2 = M'_3 - m_{12} - m_{13} - m_{21} + m_{23} - m_{31} + m_{32}.$$

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M'_3 is the new reading of the variable inductance 3. The signs of all the m's with a suffix 1 are reversed; the others are unaltered. Now reversing the primary and secondary coils of 2 we get the equation

$$M_1 - M_2 = M''_3 + m_{12} - m_{13} + m_{21} - m_{23} - m_{31} - m_{32},$$

in which the signs of all the m's with suffix 2 are reversed. Similarly, if we again reverse the primary and secondary coils of 1 we get

$$M_1 - M_2 = M'''_3 - m_{12} + m_{13} - m_{21} - m_{23} + m_{31} - m_{32}.$$

Adding up these four equations, we find that all the m's cancel out, leaving

$$M_1 - M_2 = \tfrac{1}{4}(M_3 + M'_3 + M''_3 + M'''_3).$$

Thus, in order to get the true difference between the mutual inductances of the standards 1 and 2 we have to take the mean of these four readings of 3.

After taking a reading, the connections to the alternator and the galvanometer were reversed and any changes due to this noted. In general reversal of the galvanometer connections made no difference to the balance point. There was usually a small change due to reversal of the alternator connections, and the mean of the two readings was taken as correct.

In order to verify the fact that there was no appreciable mutual inductance associated with the leads, after each series of readings the leads to the secondary coils of 1 and 2 were short-circuited and the balance obtained by varying M_3. The reading obtained was always sensibly zero.

The mean of two sets of readings taken on the same day, without disturbing the standards in any way, agreed to 0·01 μH, or one part in a million of the total mutual inductance being compared.

Before making any measurements, the standards were adjusted as follows:—

(1) The primary cylinder was levelled by means of the levelling screws on the base, a spirit level placed on the upper end of the cylinder being used to indicate when the setting was correct.

(2) The secondary coil was also levelled by means of its levelling screws, the spirit level being placed on its upper surface. In both cases the level was, of course, used in two directions at right angles.

(3) By means of inside calipers or a suitable gauge inserted between the cylindrical surface of the primary cylinder and the inner surface of the secondary ring, the secondary coil was set concentric with the primary coil. The first two adjustments made the axes of the primary and secondary coils vertical and therefore parallel. The third adjustment made them coincident.

(4) Alternating current was now passed through the two primary coils, which were connected so as to be in opposition. The vibration galvanometer was connected to the secondary coil, the height of which was adjusted by means of the levelling

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screws until the galvanometer showed no deflection. The mutual inductance between the secondary coil and each of the primary coils was then the same, and the secondary coil was therefore in the horizontal plane midway between the two primary coils. In making this adjustment it is necessary always to turn the three levelling screws by the same amount, otherwise the plane of the secondary coil will become tilted. In any case it is necessary to repeat adjustments (1) (2) and (3) to make sure that earlier adjustments are not upset by the later ones.

Adjustment (3) can also be made electrically by varying the position of the secondary coil until the mutual inductance between it and the primary winding (with the two coils connected so that their magnetic fields are additive) reaches a maximum value. In some cases the adjustment was made in this way, but the mutual inductance varies so little with change of position of the secondary coil that this method is not very convenient. The mechanical method gave identical results and was much quicker.

Both standards having been adjusted in this way, the difference between the mutual inductances was measured as described above. The calculated value of the National Physical Laboratory standard was last determined in 1914, and found to be

10, 017·78 microhenries at 15° C.

Its temperature coefficient is taken as $3{\cdot}5 \times 10^{-6}$, but it is not known with the same certainty as that of the new standard. Accordingly, the comparisons were made at a temperature of 15° C. approx. The results are most conveniently considered by using the calculated value of the new standard, and the measured difference between the new and old standards, to determine an experimental value for the Laboratory standard. Table XIV gives the results of six sets of observations made on different days. The standards were readjusted before each set of measurements. All the results are in microhenries.

TABLE XIV.—Comparison of new Japanese Standard with National Physical Laboratory Standard.

Temperature (Japanese standard.)	$M_{J.}$ (calculated).	$M_{N.P.L.}-M_{J.}$ (observed).	$M_{N.P.L.}$ (by difference).	Temperature (N.P.L. standard.)	$M_{N.P.L.}$ (corrected to 15° C.).	Deviation from mean.
°C.	μH.	μH.	μH.	°C.	μH.	μH.
15·6	10,010·03	7·81	10,017·84	15·2	10,017·84	0·02
15·0	10,010·02	7·86	10,017·88	14·3	10,017·90	0·04
15·5	10,010·03	7·84	10,017·87	15·0	10,017·87	0·01
15·0	10,010·02	7·79	10,017·81	14·2	10,017·84	0·02
16·2	10,010·04	7·83	10,017·87	15·4	10,017·86	0·00
16·2	10,010·04	7·86	10,017·90	15·9	10,017·87	0·01

Mean value of N.P.L. standard at 15° C. = 10,017·86 μH, the mean deviation from this value being 0·01₇μH.

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It will be seen that the value obtained for the National Physical Laboratory standard in terms of the new standard is

$$10{,}017{\cdot}86 \pm 0{\cdot}02 \text{ microhenries at } 15^\circ \text{ C.}$$

The value obtained by calculation in 1914 was 10,017·78 microhenries at 15° C., while the value deduced from measurements made in 1921,* using the primary standard constructed in that year as basis, was

$$10{,}017{\cdot}83 \text{ microhenries at } 15^\circ \text{ C.}$$

Thus the total discrepancy is less than 1 part in 100,000, and the difference between the 1921 and 1926 results is only three parts in a million. For all practical purposes this constitutes perfect agreement. It is concluded that the National Physical Laboratory standard has not changed appreciably in the last five years, and probably not since 1914.

* D. W. Dye, 'N.P.L. Collected Researches,' Vol. 18, Paper 7, p. 123, or 'Proc. Roy. Soc.', A, Vol. 101, p. 315, 1922.

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II.

ON THE FIELD OF FORCE NEAR THE NEUTRAL POINT PRODUCED BY TWO EQUAL COAXIAL COILS, WITH SPECIAL REFERENCE TO THE CAMPBELL STANDARD OF MUTUAL INDUCTANCE

BY

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ABSTRACT.

When a current is passed through two equal coaxial coils so that the components of the magnetic fields parallel to the axis add, there is a circle, midway between the two coils, at the circumference of which the magnetic field is zero. If a turn of wire be placed to coincide with this circle, any small variation in its radius will affect the value of the mutual induction only to the second order of small quantities. Use is made of this in the Campbell standard of mutual inductance and in F. E. SMITH's apparatus for measuring the absolute value of the ohm by the Lorenz method. The author investigates the conditions necessary to make the second order terms in the evaluation of the mutual inductance small, and the magnitudes of the changes in the mutual inductance due to displacements of the secondary in various directions. He concludes that the ratio of the diameter of the secondary to that of the primary should be made as large as is consistent with maintaining an accurately measurable size for the primary and keeping the instrument reasonably compact; also that a square section is most suitable for the secondary. In the case of the Lorenz apparatus, if the rotating discs on which the brushes are applied were bevelled at an angle of 45° to the axis any inaccuracy in the location of the brushes would affect the result only to the third order of small quantities.

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ON THE FIELD OF FORCE NEAR THE NEUTRAL POINT PRODUCED BY TWO EQUAL COAXIAL COILS, WITH SPECIAL REFERENCE TO THE CAMPBELL STANDARD OF MUTUAL INDUCTANCE.

By R. M. WILMOTTE, M.A., sometime Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Royal Society,' Series A, Vol. 107, 1925.]

Introduction and Summary.

When a current is passed through two equal coaxial coils so that the component of the magnetic fields parallel to the common axis add, there is a circle, mid-way between the two coils, at which the magnetic field is zero. At all points in the plane of that circle lying outside it, the field of force is in one direction, and at all points within the circle it is in the opposite direction. It is evident, therefore, that if a coaxial turn of wire be placed in the plane of the circle, the mutual inductance between the two coils will be a maximum when the wire coincides with the circle, and any small change in the radius of the turn will affect the value of the mutual inductance only to the second order of small quantities.

A. Campbell* developed this principle for the construction of a standard of mutual inductance. In this design two equal coaxial single layer coils formed the primary, and the secondary consisted of a large number of turns packed closely together, so that the mean circle of the secondary coincided with the neutral circle formed by the field of the two single layer coils. By this means it was not necessary to measure the diameter of the secondary coil with extreme accuracy.

The principle was also used by F. E. Smith† in the construction of an apparatus for measuring the absolute value of the ohm by the Lorenz method. A rotating disc formed the secondary of the mutual inductance with two single layer coaxial coils as the primary. Lord Rayleigh‡ had previously advocated the inverse system, in which the dimensions of the apparatus were adjusted so that a small change in the radius of the coils would have a very small effect on the value of the mutual inductance.

The accuracy of the apparatus designed on the principle first introduced by A. Campbell depends largely on the terms of the second order in the evaluation of the mutual induc-

* A. Campbell, 'N.P.L. Collected Researches,' Vol. 4, Paper 6, p. 213 (or 'Proc. Roy. Soc.,' A, Vol. 79, p. 428, 1907).

† F. E. Smith, 'N.P.L. Collected Researches,' Vol. 11, Paper 13, p. 209 (or 'Phil. Trans. Roy. Soc.,' A, Vol. 214, p. 27, 1913).

‡ Lord Rayleigh, 'Phil. Mag.,' Series 5, Vol. 14, p. 329, 1882.

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tance. The conditions necessary to make the second order terms small do not seem to have been investigated, nor has the nature of the small changes in the mutual inductance due to a displacement of the secondary in various directions been fully considered, except for the special case forming the principle of the Lorenz machine.*

These questions are investigated in this paper, and the variation of the value of the mutual inductance due to a radial displacement of the secondary is calculated and compared with the results obtained by A. CAMPBELL for a particular case.

The Variation of the Mutual Inductance due to a Small Displacement of the Secondary.

(1) *General Expression.*—Throughout this paper the units used are absolute units unless otherwise stated. Coils P and Q (Fig. 1) are equal and coaxial. These will be

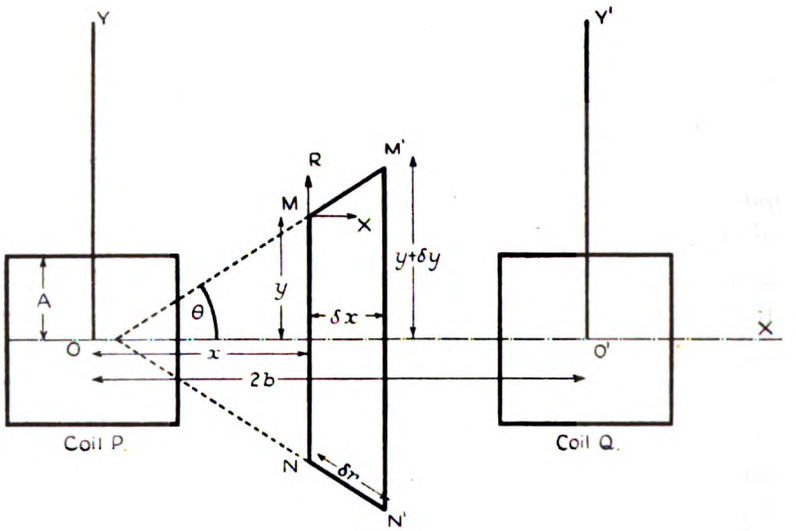

FIG. 1.

called the primary, and a single turn of radius y coaxial with the two equal coils will be called the secondary.

Let M_1 and M_2 be the mutual inductance between the single turn, situated at a distance x from the centre of coil P, and coils P and Q respectively. Suppose the secondary be moved from the position MN to the position M′ N′ where MM′ is of length δr and makes an angle θ with the axis, then

$$\delta M_1 = \left[\frac{\partial M_1}{\partial x}\,\delta x + \frac{\partial M_1}{\partial y}\,\delta y\right] + \frac{1}{\underline{2}}\left[\frac{\partial^2 M_1}{\partial x^2}(\delta x)^2 + 2\,\frac{\partial^2 M_1}{\partial x\,\partial y}\,.\,\delta x\,\delta y + \frac{\partial^2 M_1}{\partial y^2}(\delta y)^2\right]$$
$$+ \frac{1}{\underline{3}}\left[\frac{\partial^3 M_1}{\partial x^3}(\delta x)^3 + 3\,\frac{\partial^3 M_1}{\partial x^2\,\partial y}(\delta x)^2\,\delta y + 3\,\frac{\partial^3 M_1}{\partial x\,\partial y^2}\,.\,\delta x\,(\delta y)^2 + \frac{\partial^2 M_1}{\partial y^3}(\delta y)^3\right]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

* F. E. SMITH, 'N.P.L. Collected Researches,' Vol. 11, Paper 13, p. 209 (or 'Phil. Trans. Roy. Soc.,' A, Vol. 214, p. 27, 1914).

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Now the co-ordinates of M′ referred to OX, OY are $(x + \delta x,\ y + \delta y)$ and referred to O′X, O′Y′ they become $(x + \delta x - 2b,\ y + \delta y)$. Hence, by putting M_2 for M_1 and $(x - 2b)$ for x in equation (1), the corresponding equation for coil Q will be obtained. The total change in the mutual inductance to the first order of small quantities is, therefore, given by

$$\delta M_1 + \delta M_2 = \frac{\partial M_1}{\partial x}\,\delta x + \frac{\partial M_2}{\partial (x - 2b)}\,\delta x + \frac{\partial M_1}{\partial y}\,\delta y + \frac{\partial M_2}{\partial y}\,\delta y.$$

This will be zero for all values of the ratio $\delta x/\delta y$ where

$$\frac{\partial M_1}{\partial x} + \frac{\partial M_2}{\partial (x - 2b)} = 0 \quad \text{and} \quad \frac{\partial M_1}{\partial y} + \frac{\partial M_2}{\partial y} = 0.$$

Since M_1 and M_2 are similar functions, these conditions will hold when

$$x = b \quad \text{and} \quad \frac{\partial M_1}{\partial y} = \frac{\partial M_2}{\partial y} = 0. \qquad (2)$$

For these conditions, since $M = M_1 + M_2$, we have

$$\delta M = \left[\frac{\partial^2 M}{\partial x^2}(\delta x)^2 + \frac{\partial^2 M}{\partial y^2}(\delta y)^2\right] + \left[\frac{\partial^3 M}{\partial x^2\,\partial y}(\delta x)^2\,\delta y + \tfrac{1}{3}\frac{\partial^3 M}{\partial y^3}(\delta y)^3\right] + \ldots \qquad (3)$$

If X and R are the axial and radial components respectively of the magnetic force at the secondary due to coil P alone, the following relations are easily deduced

$$2\pi y\,.\,X = \frac{\partial M}{\partial y} \quad \text{and} \quad 2\pi y\,.\,R = -\frac{\partial M}{\partial x}.$$

Differentiating these relations and substituting, equation (3) becomes

$$\delta M = 2\pi y\left[\frac{\partial X}{\partial y}\,.\,(\delta y)^2 - \frac{\partial R}{\partial x}\,.\,(\delta x)^2\right] - \frac{2\pi}{3}\left[3y\,.\,\frac{\partial^2 R}{\partial x\,\partial y}\,(\delta x)^2\,\delta y + 3\frac{\partial R}{\partial x}(\delta x)^2\,\delta y - y\,.\,\frac{\partial^2 X}{\partial y^2}(\delta y)^3 - 2\frac{\partial X}{\partial y}(\delta y)^3\right] + \ldots$$

If Ω is the magnetic potential at the secondary

$$X = \frac{\partial \Omega}{\partial x} \quad \text{and} \quad R = \frac{\partial \Omega}{\partial y},$$

so that

$$\frac{\partial X}{\partial y} = \frac{\partial R}{\partial x} \quad \text{and} \quad \frac{\partial^2 X}{\partial y^2} = \frac{\partial^2 R}{\partial x\,\partial y}.$$

The final result then is

$$\delta M = 2\pi y\left\{\frac{\partial R}{\partial x}[(\delta y)^2 - (\delta x)^2] + \tfrac{1}{3}\left[\left(\frac{\partial^2 R}{\partial x\,\partial y} + \frac{2}{y}\frac{\partial R}{\partial x}\right)(\delta y)^3 - 3\left(\frac{\partial^2 R}{\partial x\,\partial y} + \frac{1}{y}\frac{\partial R}{\partial x}\right)(\delta x)^2\,\delta y\right] + \ldots\right\} \qquad (4)$$

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Putting $\delta x = \delta r \cos\theta$ and $\delta y = \delta r \sin\theta$, this becomes

$$\delta M = 2\pi y \left\{ -\frac{\partial R}{\partial x} \cdot \cos 2\theta \cdot (\delta r)^2 + \frac{\sin^3\theta}{3}\left[\frac{\partial^2 R}{\partial x\,\partial y} + \frac{2}{y}\frac{\partial R}{\partial x} - 3\cot^2\theta\left(\frac{\partial^2 R}{\partial x\,\partial y} + \frac{1}{y}\frac{\partial R}{\partial x}\right)\right](\delta r)^3 + \ldots\right\}. \quad (5)$$

(2) *The Value of* $\partial R/\partial x$ *and* $\partial^2 R/\partial x\,\partial y$ *for Single-Layer Coils.*—Consider an element of axial length δl (Fig. 2) of a single-layer coil of length L assumed to be a uniform

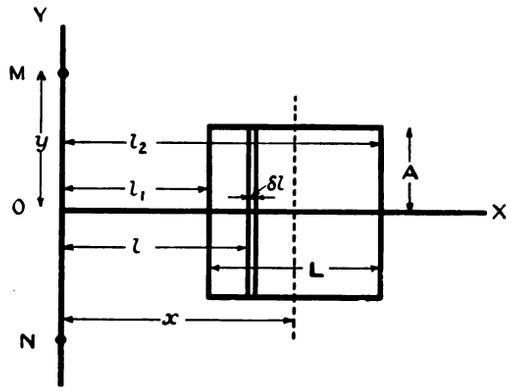

FIG. 2.

spiral of n_2 turns per unit length. The equivalence of this to a uniform current sheet was first found by VIRIAMU JONES* and later explained by Lord RAYLEIGH.† Let R be the radial component of the magnetic force at a point M due to a unit current in the helix, which is situated with its mean plane at a distance x from O. Then

$$R = \int_{x-L/2}^{x+L/2} n_2 R'\,dl,$$

where R′ is the radial component of the magnetic force due to a single turn. Differentiating, we have

$$\frac{\partial R}{\partial x} = \left[n_2 R'\right]_{l_1}^{l_2} \quad \text{and} \quad \frac{\partial^2 R}{\partial x\,\partial y} = \left[n_2 \frac{\partial R'}{\partial y}\right]_{l_1}^{l_2}.$$

Similarly

$$X = n_2 \int_{l_1}^{l_2} X'\,dl.$$

If there are n_1 turns on the secondary, equation (5) becomes

$$\delta M = -2\pi y n_1 n_2 \left\{\cos 2\theta \left(R'\right)_{l_1}^{l_2} (\delta r)^2 - \frac{\sin^3\theta}{3}\left[\left(\frac{\partial R'}{\partial y}\right)_{l_1}^{l_2} + \frac{2}{y}\left(R'\right)_{l_1}^{l_2} - 3\left(\frac{\partial R'}{\partial y}\right)_{l_1}^{l_2}\cot^2\theta - \frac{3}{y}\left(R'\right)_{l_1}^{l_2}\cot^2\theta\right](\delta r)^3 + \ldots\right\}, \quad (6)$$

* J. VIRIAMU JONES, 'Proc. Roy. Soc.,' A, Vol. 63, p. 192, 1898.

† Lord RAYLEIGH, 'B.A. Report,' p. 241, 1899.

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and the condition given by equation (2) becomes

$$\int_{l_1}^{l_2} X' \, dl = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

ALEXANDER RUSSELL* has shown that, for a coil of radius A,

$$R' = \frac{2}{r_1}\left[\frac{2Al}{r_2^2}E - \frac{l}{y}(F - E)\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$X' = \frac{2}{r_1}\left[\frac{2A\,(A - y)}{r_2^2}E + (F - E)\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $r_1^2 = (A + y)^2 + l^2$ and $r_2^2 = (A - y)^2 + l^2$, E and F being the complete elliptic integrals to the modulus $\sqrt{1 - r_2^2/r_1^2}$ of the first and second kinds.

By means of these equations, the curves in Figs. 3 and 4 were calculated, taking the diameter of the primary as unity. These curves can be applied to any value of the diameter of the primary, since the magnetic force is inversely proportional to the linear dimensions of the coils.

The arithmetico-geometric means applied by L. V. KING† for calculating the complete elliptic integrals can be used with convenience for obtaining these curves, if a calculating machine is available.

Since the terms of the third order are small, it is not necessary to evaluate $\partial R'/\partial y$ to any great accuracy. This can be obtained by estimating the gradient of the curves obtained by plotting R′ against $y/2A$, from Fig. 3, at the required points. Should greater accuracy be required, it would be necessary to differentiate equation (7) with respect to y, remembering that, if k is the modulus,

$$\frac{\partial E}{\partial k} = \frac{1}{k}(E - F) \qquad \text{and} \qquad \frac{\partial F}{\partial k} = \frac{E}{k(1 - k^2)} - \frac{F}{k}.$$

The relation obtained is rather cumbersome. Fortunately, the effect of the third-order terms is negligible in the calculation of the standard apparatus to which this paper mainly applies.

If the section of the secondary is finite, the variation of the value of the mutual inductance from the ideal case, when all the turns of the secondary are concentrated at the neutral circle, can be obtained by integrating equation (4) over the whole section of the winding. A. CAMPBELL‡ and D. W. DYE§ evaluated the effect from a formula of

* A. RUSSELL, 'Proc. Phys. Soc. Lond.,' Vol. 20, p. 478, 1907.

† L. V. KING, 'Proc. Roy. Soc.,' A, Vol. 100, p. 60, 1921.

‡ A. CAMPBELL, 'N.P.L. Collected Researches,' Vol. 9, Paper 6, p. 49 (or 'Proc. Roy. Soc.,' A, Vol. 87, p. 394, 1912).

§ D. W. DYE, 'N.P.L. Collected Researches,' Vol. 18, Paper 7, p. 122 (or 'Proc. Roy. Soc.,' A, Vol. 101, p. 315, 1922).

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G. F. C. SEARLE deduced for a general case of a coil having a rectangular cross-section in any field symmetrical about the axis of the coil. To obtain a similar expression from

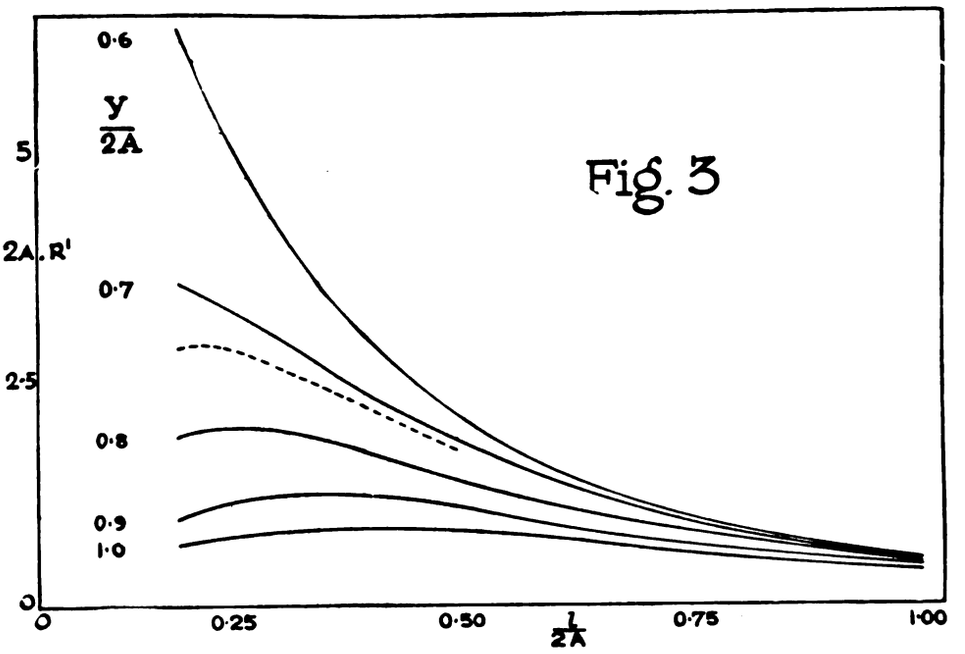

Fig. 3

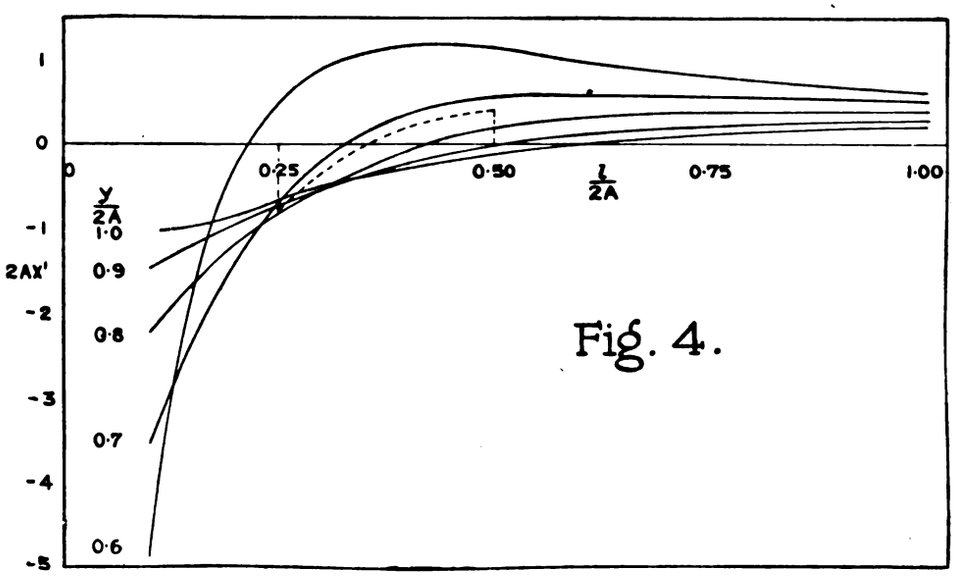

Fig. 4.

equation (4) it is necessary to expand the series to one other term, when a formula equivalent to that of G. F. C. SEARLE will be obtained.

(3) *Comparison with the Campbell Standard of Mutual Inductance.*—The dotted lines in Figs. 3 and 4 correspond to the design of the Campbell standard of mutual inductance at present in the National Physical Laboratory. A. CAMPBELL calculated the value of

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$\partial M/\partial y$, using the following formula obtained by differentiating the expression deduced by VIRIAMU JONES* for the mutual inductance of a spiral and a circle

$$\frac{\partial M}{\partial y} = \frac{n_1 n_2 \sqrt{Ay}}{A(A+y)}\left[2Ak F + \frac{(A+y)\sqrt{Ay}}{l}\psi\right]_{l_1}^{l_2}, \quad \ldots\ldots \quad (10)$$

where ψ is the complete elliptic integral of the third kind and n_2 is the number of turns per unit length of the spiral. The values used were $A = 10$, $l_1 = 5$, $l_2 = 10$, $n_1 n_2 = 10{,}000$.

From the curve obtained A. CAMPBELL deduced that

$$\frac{\partial M}{\partial y} = 0 \qquad \text{when} \qquad y = 14{\cdot}583,$$

and

$$\frac{\partial M}{\partial y} = -\,0{\cdot}1077\,\xi + 0{\cdot}016\xi^2 + \ldots,$$

where M is measured in millihenries and $\xi = y - 14{\cdot}583$.

From Fig. 3 it will be seen that

$$\left[2AR'\right]_{0{\cdot}25}^{0{\cdot}5} = 1{\cdot}17 \qquad \text{and} \qquad \left[4A^2\frac{\partial R'}{\partial y}\right]_{0{\cdot}25}^{0{\cdot}5} = 10{\cdot}2.$$

Substituting the above values in equation (5) and putting $\theta = \pi/2$, we have

$$\delta M = -\,0{\cdot}054\,(\delta r)^2 + 0{\cdot}0053\,(\delta r)^3 + \ldots$$

in which M is measured in millihenries. On differentiating this becomes

$$\frac{\partial M}{\partial r} = -\,0{\cdot}108\delta r + 0{\cdot}016\,(\delta r)^2 + \ldots.$$

This agrees with the result obtained by A. CAMPBELL. It will be also seen that the area enclosed by the dotted line in Fig. 4 is zero, so that the condition required by equation (7) holds.

If the terms of the third and higher orders are neglected, as was found admissible by A. CAMPBELL† and D. W. DYE‡ in the calculation of this standard of mutual inductance,

* J. VIRIAMU JONES, *loc. cit.*

† A. CAMPBELL, 'N.P.L. Collected Researches,' Vol. 9, Paper 6, p. 49 (or 'Proc. Roy. Soc.,' A, Vol. 87, p. 394, 1912).

‡ D. W. DYE, 'N.P.L. Collected Researches,' Vol. 18, Paper 7, p. 122 (or 'Proc. Roy. Soc.,' A, Vol. 101, p. 315, 1922).

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δM varies as $\cos 2\theta$. This result is shown graphically in Fig. 5, in which polar co-ordinates are used, the radius being proportional to δM.

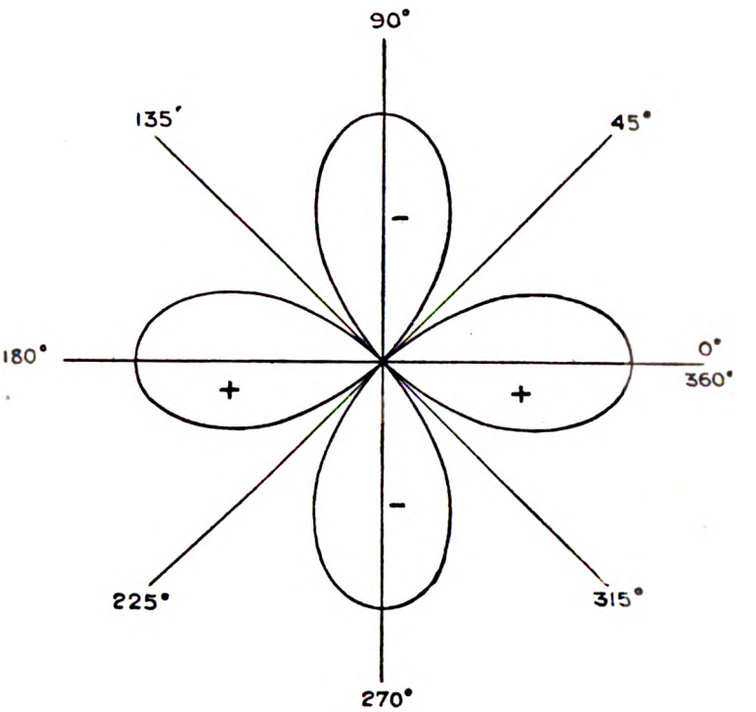

FIG. 5.

(4) *Method of Finding the Neutral Circle.*—Suppose the ratio $y/2A$ is known, the approximate value of l_1 and l_2 which will make $\int_{l_1}^{l_2} X'\,dl = 0$ can be found from Fig. 4. The accurate value of this integral can then be calculated from equation (10), remembering that

$$\frac{\partial M}{\partial y} = 2\pi y \int_{l_1}^{l_2} X'\,dl.$$

This can be made zero by adding $2\pi y X_2' \delta l_2$, where X_2' is the axial magnetic force due to a unit circular current at l_2, and can be read off from the curve in Fig. 4. The value of δl_2 can then be found and added to l_2, giving a second approximation to the dimensions of the primary coil. The process can be repeated until the necessary accuracy is attained.

Conclusion.

It has been shown that, to keep the second-order terms small, it is necessary to make $\left[R'\right]_{l_1}^{l_2}$ small, still retaining the condition given by equation (7). By examining the curves of Figs. 3 and 4 it will be seen that this can be done by making the length of the

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single-layer primary coils small, and the ratio of the diameters of the secondary and primary as large as possible.

It is not advisable to reduce the diameter of the primary in order to increase the ratio $y/2A$, for the accuracy in the measurement of the dimensions of the primary would thereby be decreased. On the other hand, it is not possible to increase the diameter of the secondary to any great extent, for the condition required by equation (7) shows that the distance between the primary and the secondary would have to be increased, rendering the apparatus very cumbersome and requiring a greater number of turns to give the same mutual inductance. This is a disadvantage, for the correction due to the self-capacity of the secondary becomes very important. The curves show, however, in which direction an improvement can be made, the best design depending on the kind of measurement to be made, and the maximum size of the apparatus convenient from a mechanical point of view.

The best shape for the secondary can be seen from Fig. 5. A displacement of the secondary at an angle of 45° with the axis produces a change in mutual inductance only to the third order of small quantities. This could have been deduced *a priori*, for the equi-potential surface which cuts itself must do so in this case at an angle of 90°. It is not, however, usually practicable to use a complex shape for the section of the coil, for the new errors introduced would balance any advantage gained. A square section will, in general, be found most convenient.

In the case of the Lorenz apparatus, a suggested refinement, as the outcome of the deductions of this paper, might be to bevel the rotating discs on which the brushes are applied at an angle of 45° to the axis, so that any inaccuracy in the location of the brushes would only affect the result to the third order of small quantities.

In conclusion I wish to express my thanks to Dr. G. F. C. SEARLE for his kind criticism.

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III.

A PRECISION METHOD FOR THE COMPARISON OF UNEQUAL MUTUAL INDUCTANCES AT TELEPHONIC FREQUENCIES

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ABSTRACT.

A new method for the comparison of unequal mutual inductances at telephonic frequencies is described. It is considered chiefly as a means of calibrating high-range mutual inductance standards. Its advantages are :—

(1) The instrument can be tested with any desired common point.

(2) The " impurity " or " phase defect " of the instrument is determined at the same time as the mutual inductance.

Results of tests on a 100-millihenry inductometer are given in order to show the importance of the common point on mutual inductances used with current of telephonic frequency.

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A PRECISION METHOD FOR THE COMPARISON OF UNEQUAL MUTUAL INDUCTANCES AT TELEPHONIC FREQUENCIES.

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Introduction.

In this country measurements of inductance and capacity are generally made in terms of a standard of mutual inductance, the fundamental standard for such measurements being the Campbell mutual inductance standard at the National Physical Laboratory. The comparison of two mutual inductances is thus a measurement of considerable importance. The methods hitherto available for making such measurements are discussed by A. Campbell in Glazebrook's 'Dictionary of Applied Physics.'* For the comparison of unequal mutual inductances two methods are available, those of Maxwell and A. Campbell respectively. From the point of view of alternating current measurements Maxwell's method suffers from the disadvantage that it requires a knowledge of the effective resistance of the secondary coils of the mutual inductances. Campbell's method avoids this difficulty, but, in common with Maxwell's method, it only holds accurately for perfectly pure mutual inductances, *i.e.*, both these methods assume that the secondary induced voltage is exactly 90° out of phase with the primary current. In practice this is known to be only approximately the case. At the higher telephonic frequencies, the phase angle between the secondary voltage and primary current is always appreciably different from 90°. Let this phase angle be ϕ. We shall call the angle $\delta = 90° - \phi$ the phase defect of the mutual inductance. It is a measure of the "impurity." The vector relation connecting secondary induced voltage E_s and primary current I_p may be expressed by the equation

$$E_s = (\sigma + jMw)\, I_p,$$

where M is the mutual inductance, $w/2\pi$ is the frequency, and j is the operator rotating through a right angle. This equation is used by Butterworth. σ is called the impurity† of the mutual inductance. It is of the dimensions of a resistance, and is connected with the phase defect δ by the relation

$$\tan \delta = \frac{\sigma}{Mw}.$$

* Glazebrook, 'Dictionary of Applied Physics,' Vol. 2, p. 424. (Macmillan, London, 1922.)

† See Silsbee, 'Bull. Bureau of Standards,' Vol. 13, Bull. 281, p. 414, 1916.

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BUTTERWORTH* has made a mathematical investigation of the effect on mutual inductances of the self and mutual capacities of the coils and the eddy current losses in them. He shows, among other things, that these cause the effective mutual inductance to vary with the frequency, the changes being proportional to the square of the frequency, and that they also introduce an impurity σ. He also points out that since the mutual capacities between the two coils may be very different when different points on the coils are made common, as in the Carey Foster bridge and the Campbell frequency bridge, the actual value of the mutual inductance may be different for the same instrument used in different bridges. If the mutual inductance is used with no common point, then the relative potentials of the primary and secondary coils are probably largely determined by stray capacities. Thus the mutual capacity effect, and therefore the value of the mutual inductance, are liable to be uncertain, and to vary during the course of an experiment. Thus it is desirable to have a common point on primary and secondary in all cases where mutual inductances are used with high-frequency current.

In attempting to use the Maxwell or Campbell methods for the comparison of large mutual inductances at high frequencies, the following difficulties are encountered:

(*a*) No true balance is obtained, but only a minimum of sound in the telephone. This is due to the fact that the impurity terms are not balanced.

(*b*) The reading given by the minimum of sound is liable to change, since the relative potentials of primary and secondary circuits are not accurately defined by using common points. In the method to be described, these difficulties are avoided. Common points are used, and the impurity terms are balanced, so that perfect silence is obtained in the telephone. Further, if one of the instruments to be compared is a standard of known impurity, the method permits of the impurity of the other instrument being determined.

The Convention as to Sign.

The direction of the induced E.M.F. in a coil is reversed by changing the direction of the winding. Thus we may regard a mutual inductance as being either positive or negative. Similarly the impurity term may be either positive or negative, and it is necessary to establish a convention with regard to the sign to be taken. BUTTERWORTH* takes the mutual inductance M as positive when the directions of winding of the two coils are such that (one point being common), if current is applied to the combination at the open ends, the mutual inductance will oppose the self-inductances. We shall adopt the same convention but express it in a different way. Let Fig. 1 (*a*) represent the two coils. Let I_p be the primary current flowing towards the common point as shown, and let E_s be the secondary voltage, also reckoned as acting towards the common point. Then when E_s leads I_p, as shown in the vector diagram Fig. 1 (*b*),

* S. BUTTERWORTH, 'N.P.L. Collected Researches,' Vol. 18, Paper 5, p. 41; or 'Proc. Phys. Soc., Lond.,' Vol. 33, p. 312, 1921.

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the mutual inductance is regarded as positive. If σ is also positive, we see that the secondary voltage leads the primary current by an angle ϕ less than 90°, and the phase defect δ is also positive.

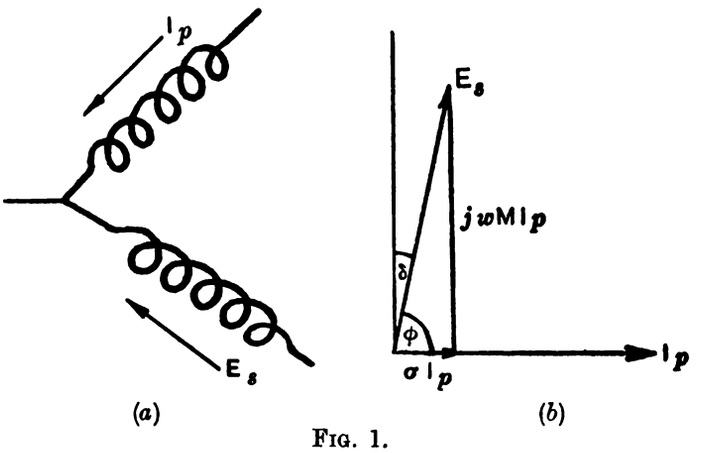

FIG. 1.

The vector diagram will be unaltered if the primary current and secondary voltage are reckoned as both acting away from the common point, instead of both acting towards it, but in order to keep to the present convention one of these alternatives must be chosen. The equation

$$E_s = (\sigma + jMw)\, I_p$$

may then be applied.

Description of Method.

The method is shown in Fig. 2. M_1, M_2 are the mutual inductances, R_1, R_2 are non-inductive resistances (the usual type of ratio arms), L is a variable self-inductance, and R_4 a variable non-inductive resistance ; r_3 is a small resistance, a slide wire with a

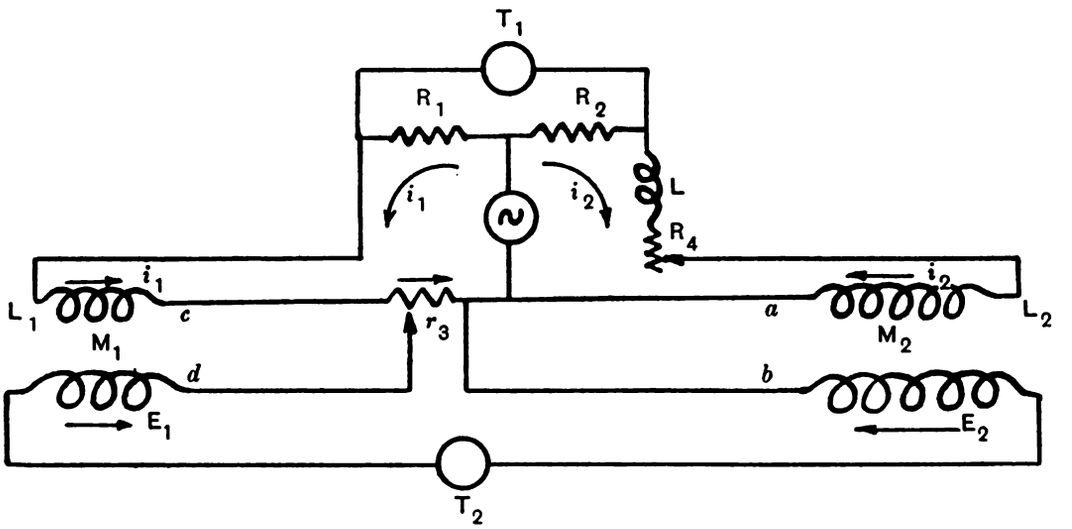

FIG. 2.

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sliding potential point, T_1 and T_2 are telephones. The coils L_1 and L_2 are the primaries of the mutual inductances. The telephone T_2 is first switched off, leaving the secondary circuit open. The primary circuit (really a self-inductance bridge) is then balanced by adjusting L and R_4 until no sound is heard in T_1. The sole purpose of L and R_4 is to enable this adjustment to be made. When balance is obtained, let i_1 and i_2 be the currents in the two primaries in the directions shown. Then it is easy to see that since the points A and B are at the same potential we must have

$$R_1 i_1 = R_2 i_2. \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Now let T_2 be switched into the secondary circuit, and vary the tapping point on r_3 and the value of M_1 until T_2 becomes silent. When the balance is obtained there is still no current in the secondary circuit and thus the primary currents i_1 and i_2 are unaltered. The E.M.F.'s acting in the secondary circuit are the two secondary E.M.F.'s E_1 and E_2 acting in the directions shown, in accordance with the above-mentioned convention, and the potential difference between the two potential points on r_3. If r_3 is the resistance between these two points, this potential difference is $r_3 i_1$ and acts in the direction of i_1. Thus the vector equation for the second balance is

$$E_1 + i_1 r_3 - E_2 = 0.$$

This may be written

$$(\sigma_1 + jwM_1) i_1 + i_1 r_3 - (\sigma_2 + jwM_2) i_2 = 0, \quad \ldots\ldots\ldots (2)$$

which, by (1), becomes

$$(\sigma_1 + jwM_1) R_2 + r_3 R_2 - (\sigma_2 + jwM_2) R_1 = 0.$$

Thus, equating the rectangular components,

$$M_1 R_2 - M_2 R_1 = 0, \quad \ldots\ldots\ldots\ldots (3)$$

$$(\sigma_1 + r_3) R_2 = \sigma_2 R_1. \quad \ldots\ldots\ldots\ldots (4)$$

Equation (3) gives the unknown mutual inductance M_2 in terms of the standard M_1, and (4) gives the impurity σ_2 of the unknown mutual inductance in terms of σ_1, that of the standard, and the reading r_3.

In the case of standard inductances the term σ_2 is very small and, in order to determine it accurately, it becomes necessary to take into account the fact that the ratio arms R_1 and R_2 possess small residual inductances l_1 and l_2. In this case, instead of equation (1), we get

$$i_1 (R_1 + jl_1 w) = i_2 (R_2 + jl_2 w).$$

Substituting this in (2) and equating rectangular components as before, we get

$$M_1 R_2 - M_2 R_1 + l_2 (r_3 + \sigma_1) - l_1 \sigma_2 = 0 \quad \ldots\ldots\ldots (5)$$

and

$$(r_3 + \sigma_1) R_2 - \sigma_2 R_1 - l_2 M_1 w^2 + l_1 M_2 w^2 = 0. \quad \ldots\ldots\ldots (6)$$

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From (5) we see that the relation $M_1R_2 = M_2R_1$ holds so long as small quantities of the second order can be considered negligible. Equation (6) can be thrown into more convenient form by substituting in it this relation

$$M_1R_2 = M_2R_1.$$

We get

$$\frac{\sigma_2}{M_2w} = \frac{\sigma_1}{M_1w} + \frac{r_3}{M_1w} + \frac{l_1w}{R_1} - \frac{l_2w}{R_2},$$

i.e.,

$$\delta_2 = \delta_1 + \frac{r_3}{M_1w} + \alpha_1 - \alpha_2, \quad \ldots\ldots\ldots\ldots \quad (7)$$

where δ_2 and δ_1 are the phase defects of the two mutual inductances, and α_1, α_2 are the phase angles of the two ratio arms R_1 and R_2. This relation also holds so long as small quantities of the second order are negligible. Obviously, if the ratio arms are adjusted so that $\alpha_1 = \alpha_2$, the correction due to their residual inductances vanishes.

When the circuit is as shown in Fig. 2, the points *a* and *b* are common on inductometer M_2 and *c* and *d* are common on M_1. If the effect of making other points common is to be investigated, it is merely necessary to make the appropriate reversals of the primary or secondary coils. When inductometers possessing one fixed coil and one subdivided coil are to be tested, the fixed coil should be placed in the primary circuit, so that the T_1 balance is practically unchanged during the complete test of the instrument. Inductometers of the Campbell type usually have a tapping brought out from the middle point of the primary, and in self-inductance measurements this point is connected to a point

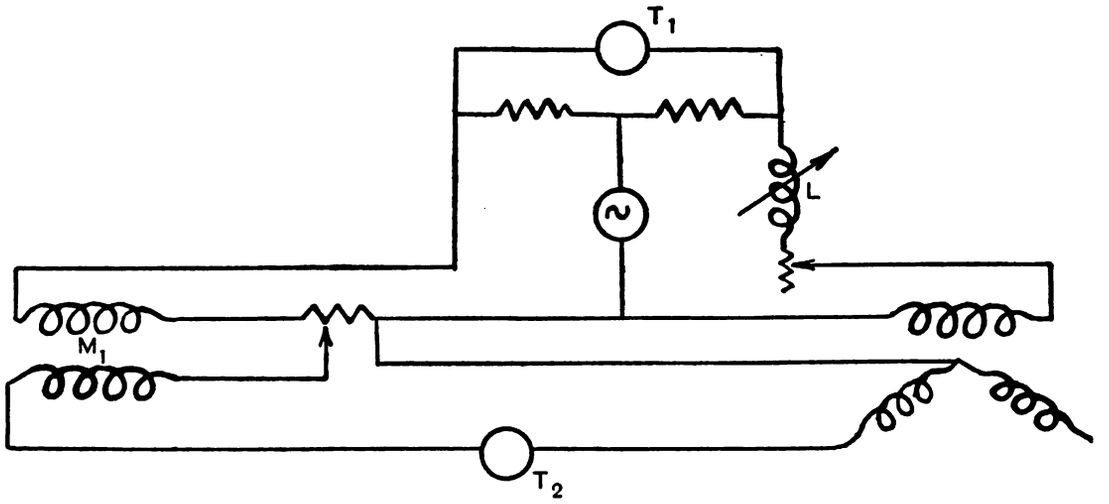

FIG. 3.

on the secondary. The method may be adapted for testing inductometers under this condition. The arrangement is shown in Fig. 3. The mutual inductance and impurity between the primary and each half of the secondary are separately measured exactly

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as before. If m_1, σ_1 and m_2, σ_2 are the values for the two halves, then for the complete instrument

$$M = m_1 - m_2, \qquad \sigma = \sigma_1 - \sigma_2.$$

The negative sign appears owing to the fact that, with the convention adopted, the mutual inductances between the primary and the two halves of the secondary are of opposite sign. In Fig. 4, I_p represents the primary current, E_1 and E_2 represent the

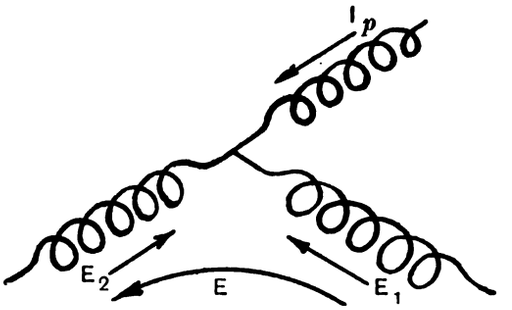

FIG. 4.

E.M.F.'s in the two halves of the secondary, and E the resultant secondary E.M.F. Obviously

$$\begin{aligned} E &= E_1 - E_2 \\ &= I_p [\sigma_1 + jm_1 w - \sigma_2 - jm_2 w] \\ &= I_p [\sigma + jMw]. \end{aligned}$$

Equating the two components, we obtain the relations given above.

If the two mutual inductances to be compared are equal—if, for example, an inductometer is to be calibrated against a standard of the same value—the above method is

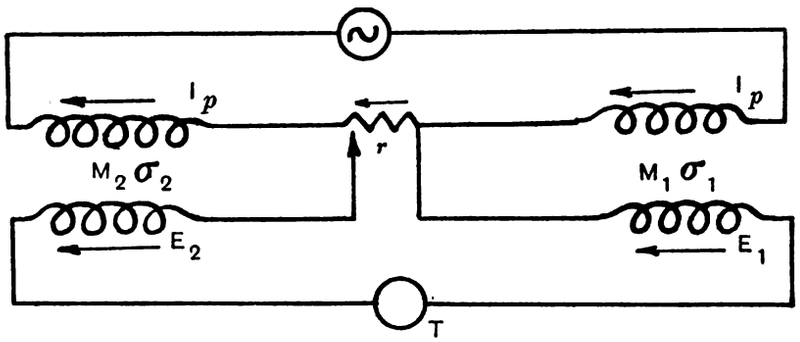

FIG. 5.

not necessary. The arrangement adopted is shown in Fig. 5. In this case the same current I_p passes through both primaries. The vector equation for balance is

$$(\sigma_1 + jM_1 w) I_p + rI_p + (\sigma_2 + jM_2 w) I_p = 0,$$

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from which we obtain as the condition for balance

$$\sigma_1 + r + \sigma_2 = 0, \quad M_1 + M_2 = 0.$$

In this case the two mutual inductances must be of opposite sign. In the method for unequal mutual inductances they must be of the same sign. In either case it may be necessary to reverse the potential points on the resistance r in order to get balance. In such cases the sign of r in the equation for balance is changed.

Sources of Error.

The sources of error are those common to all inductance bridges, *i.e.*,

(1) Cross mutual inductance effects between the coils in various arms of the bridge.

(2) Earth capacity effects.

(3) Effects of the capacities of the leads on the constants of the inductances to which they are connected.

(4) Induction effects between the telephone, source of current, and the various coils in the bridge arms.

Cross mutual inductance effects between the two inductometers are avoided by placing the two instruments sufficiently far apart. This necessitates the use of fairly long leads. The capacity of these leads will affect the values of the constants of the inductometers and this effect must not be lost sight of. The actual changes caused by leads of given capacity can be calculated by means of the formulæ given by BUTTERWORTH.* In general the capacities of the leads should be kept as small as possible, and the various coils should be so disposed that undesirable induction effects are reduced to a minimum. Leakage and capacity to earth should be kept small by supporting the instruments on paraffin wax blocks. The source of current should also be placed as far from the bridge as possible and the current brought to the bridge through bifilar leads, preferably with an earthed lead sheath. An idea of the residual errors due to induction effects between source and telephone, and to earth capacity effects, is obtained by taking readings after reversing the source and telephone-connecting leads, and also after changing the position of the telephone T_2 in the secondary circuit or earthing one of its terminals.

Examples.

Using the method here described, a detailed study of the properties of inductometers at telephonic frequency has been made and forms the subject of the next paper.† In

* S. BUTTERWORTH, *loc. cit.*

† L. HARTSHORN, 'N.P.L. Collected Researches,' Vol. 21, Paper 4, p. 55; or 'Proc. Phys. Soc., Lond.,' Vol. 38, p. 302, 1926.

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order to illustrate the importance of calibration under proper conditions, the following values are here given for an inductometer of the Campbell type, of range 100 millihenries. The instrument is shown diagrammatically in Fig. 6. DB is the primary with the middle point A brought out. FE is the secondary, comprising two decade dials and a moving coil. The instrument was tested against a 10-millihenry inductometer, the properties of which had been determined by BUTTERWORTH'S method. The results given in the table show that, whereas changes of common point have little effect at 100 cycles per sec., at 900 cycles the effects are considerable, and, if the instrument is to be used at telephonic frequencies it is essential that it should be calibrated for each of the conditions under which it is to be used.

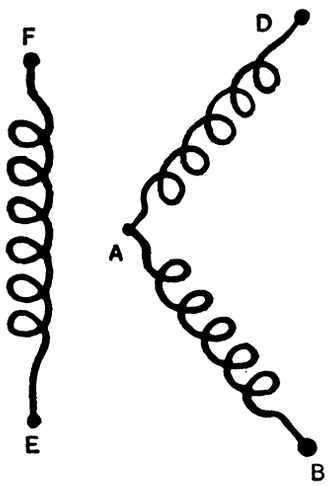

FIG. 6.

Effective Mutual Inductance of Campbell Inductometer (100 mH) *under different Conditions.*

Inductometer reading.	Frequency.	Common point.	Actual mutual inductance.
mH.	cycles per sec.		mH.
100	100	AE	100·08
		AF	100·08
		DF	100·08
		DE	100·09
		BF	100·07
		BE	100·07
100	900	AE	100·93
		AF	101·06
		DF	101·22
		DE	100·89
100	1800	AE	103·44

A preliminary measurement of the impurity of this instrument with common point DE gave a value of 0·6 ohm at 1,000 cycles. This corresponds to a phase defect of nearly 0·001 radian or about 3 minutes. Thus, under these conditions, the phase angle of the instrument is of the same order as that of a condenser with a power factor of 0·001.

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IV.

THE PROPERTIES OF MUTUAL INDUCTANCE STANDARDS AT TELEPHONIC FREQUENCIES

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ABSTRACT.

It is well known that when a mutual inductance carries alternating current, the value of the effective mutual inductance varies with the frequency, and further that the secondary P.D. is not exactly in quadrature with the primary current. Thus certain frequency corrections are introduced into all alternating-current bridges in which a mutual inductance standard is used. An experimental investigation has been made of the actual magnitude of such corrections for mutual inductance standards of three types :—

(1) A stranded wire fixed standard, with spaced winding to minimise capacity effects.

(2) The Campbell variable mutual inductance standard.

(3) The Tinsley variable mutual inductance standard made in accordance with BUTTERWORTH'S recommendations.

Values of the corrections are given for all these instruments under various conditions. It is shown that, whereas the variation of mutual inductance with frequency is mainly due to capacity effects, dielectric losses in the insulation or alternating current conductance play an important part in the determination of the phase defect, the impurity being found to be proportional to a power of the frequency higher than the second, over the telephonic range of frequency.

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THE PROPERTIES OF MUTUAL INDUCTANCE STANDARDS AT TELEPHONIC FREQUENCIES.

By L. HARTSHORN, B.Sc., A.R.C.S., D.I.C., Senior Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 38, 1926.]

Introductory.

Mutual inductance standards may be used in alternating-current bridge work for making a great variety of measurements. They may be used for measuring capacity or inductance, in which case they may be regarded as standards of reactance. They may also be used in potentiometer methods for measuring phase angles, *e.g.*, the power factor of a condenser, or the phase angle of a standard shunt. In such cases they may be regarded as standards of phase angle.

In using a mutual inductance as a standard of phase angle we make use of the fact that when alternating current passes through the primary coil, the E.M.F. induced in the secondary coil is exactly in quadrature with the current in the primary coil. The quantity we are concerned with in bridge and potentiometer measurements is not so much the E.M.F. induced in the secondary coil, as the potential difference between its terminals. In a perfect mutual inductance the P.D. between the terminals of the secondary coil on open circuit would be equal and opposite to the E.M.F. induced in that coil, but in practice, owing to the existence of capacity and eddy-current effects in the secondary coil, and also capacity effects between the primary and secondary coils, the P.D. between the secondary-coil terminals is not exactly equal and opposite to the induced E.M.F., but differs somewhat from this both in magnitude and phase. Further, as mentioned above, the E.M.F. induced in the secondary coil by a definite current flowing through the primary coil is exactly in quadrature with that current, but owing to distributed capacity and eddy-current effects in the primary coil and between the two coils, the current is not exactly the same in every element of the primary coil. The current we measure in practice is the current flowing into one primary terminal and out of the other. If earth-capacity effects are not suitably distributed these two currents will not be exactly equal, but here earth-capacity effects will not be considered, since they are not so much a property of the instrument itself as of its position and potential distribution. Accordingly, we shall consider the primary current I_p as flowing into one primary terminal and out of the other. It is obvious from what has been said that the P.D. (V_s) between the secondary terminals on open circuit will not usually be in quadrature with I_p. We may write

$$V_s = (\sigma + j\mathrm{M}\omega)\, I_p, \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

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where M is the effective mutual inductance, σ is the impurity, $\omega/2\pi$ = frequency, and j is the operator rotating through a right angle. We may regard $\theta = \tan^{-1}(M\omega/\sigma)$ as being the phase angle of such a mutual inductance. It is very nearly equal to 90°. The small angle $\delta = 90° - \theta = \tan^{-1}(\sigma/M\omega)$ may be regarded as the phase defect of the instrument. This paper gives an account of an experimental investigation into the properties of actual mutual inductance standards of various types, with special reference to the impurity and change of effective mutual inductance at the higher telephonic frequencies.

General Theoretical Considerations.

BUTTERWORTH* has investigated mathematically the effect of eddy currents and capacity effects on the effective mutual inductance and impurity of two coils. Let AB, EF (Fig. 1) be the two coils. As an approximation the distributed-capacity effects in the system are represented by the capacities k_1, k_2, k_3, k_4, k_5, k_6, which are regarded as concentrated at the ends of the coils. In order to simplify the problem further, suppose the two coils have a common point, say AE. The capacity k_3 is now short-circuited and disappears, k_1 and k_6 are thrown in parallel and may be represented by a single capacity C_1, k_2 and k_5 similarly form the single capacity C_2, and we have the system of Fig. 2, in which $C_{12} = k_4$. BUTTERWORTH shows that this system possesses the following properties.

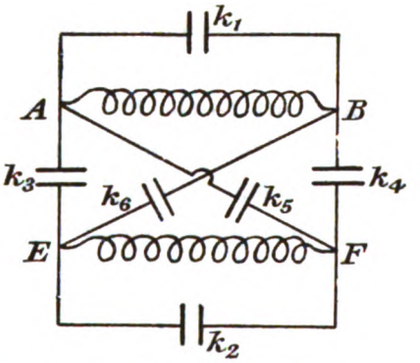

FIG. 1.—Equivalent Network for Mutual Inductance Standard.

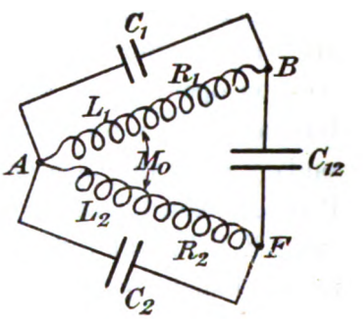

FIG. 2.—Simplified Network when the Two Coils have one Point Common.

Let L_1R_1 be the resistance and self-inductance of the primary coil, L_2R_2 those of the secondary coil, and M_0 the true or geometrical mutual inductance between them, then the effective mutual inductance and impurity as defined by equation (1) are given by

$$M = M_0 + C_{12}R_1R_2 + \omega^2\{C_1L_1M_0 + C_2L_2M_0 - C_{12}(L_1 - M_0)(L_2 - M_0)\} \quad . \quad (2)$$

$$\sigma = \omega^2[C_1R_1M_0 + C_2R_2M_0 - C_{12}\{R_1(L_2 - M_0) + R_2(L_1 - M_0)\}]. \quad . \quad . \quad . \quad . \quad (3)$$

* S. BUTTERWORTH, 'N.P.L. Collected Researches,' Vol. 18, Paper 5, p. 41; or 'Proc. Phys. Soc. Lond.,' Vol. 33, p. 312, 1921.

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These expressions only hold when the capacity effects are comparatively small, *i.e.*, when terms such as $C_1\omega$ are small compared with those such as $1/L_1\omega$, small quantities of the second order being negligible. Further, the sign of M_0 may be either positive or negative. If the direction of winding of the two coils is such that when current flows in at one open end, through the coils, and out of the other, the mutual inductance opposes the self-inductance, M_0 must be taken as positive. If the mutual inductance assists the self-inductance under these conditions, M_0 must be taken as negative. We shall accordingly speak of positive and negative mutual inductances. In order that this definition may be consistent with equation (1), we must suppose that the secondary potential difference V_s is reckoned in a definite direction, viz., when I_p flows towards the common point, V_s is also directed towards the common point (see Fig. 3), *i.e.*, in the diagram F is considered to be at a higher potential than A. If the secondary coil is contained in a closed circuit, so that the induced E.M.F. produces a current, this current will flow from F to A in the part of the circuit external to the coil, and from A to F through the coil, *i.e.*, the E.M.F. induced in the coil must be regarded as acting in the direction AF, as shown by the dotted arrow.

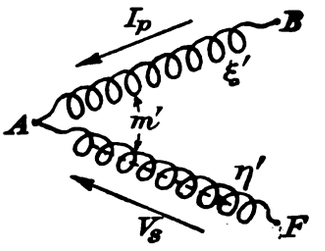

FIG. 3.—Equivalent System of Impedance Operators.

Equation (2) gives the change of effective mutual inductance with frequency as a result of capacity effects. At very low frequencies the effective mutual inductance is equal to the geometrical mutual inductance M_0, since the term involving ω^2 is then negligible, while the term $C_{12}R_1R_2$ is nearly always negligible. As the frequency increases, M increases by an amount

$$\Delta M = \omega^2 \{C_1L_1M_0 + C_2L_2M_0 - C_{12}(L_1 - M_0)(L_2 - M_0)\}. \quad . \quad . \quad . \quad . \quad (4)$$

For convenience $\Delta M/M_0$ is often referred to as the frequency coefficient of the mutual inductance. Thus

$$\text{Frequency coefficient} = \frac{\Delta M}{M_0} = \omega^2 \left\{C_1L_1 + C_2L_2 - C_{12} \cdot \frac{(L_1 - M_0)(L_2 - M_0)}{M_0}\right\}. \quad (5)$$

We may regard $L_1C_1\omega^2$ as the frequency coefficient due to the self-capacity of the primary coil, and $L_2C_2\omega^2$ that due to the self-capacity of the secondary coil. Both these terms are always positive. The frequency coefficient due to the mutual capacity between the two coils is $-C_{12}\frac{(L_1 - M_0)(L_2 - M_0)\omega^2}{M_0}$, and this term may be either positive or negative, depending on the relative value of L_1, L_2 and M_0. When M_0 is negative, this term is always positive, and is numerically equal to $C_{12}\frac{(L_1 + M_0)(L_2 + M_0)}{M_0}$. When M_0 is positive, and $L_1 > M_0$, $L_2 > M_0$, which is commonly the case in practice,

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the term is negative, and thus the frequency coefficient due to mutual capacity opposes that due to self-capacity. If $L_1 > M_0 > L_2$, the term is positive, but is numerically much smaller than the case in which M_0 is negative, provided C_{12} is the same in the two cases. Thus a positive mutual inductance has a much smaller frequency coefficient than a negative one, the capacities being the same in the two cases. This is an important practical point, since a mutual inductance standard can readily be changed from positive to negative by changing the common point,and in some bridges M may be of either sign. In such cases it is nearly always preferable to use a positive M.

Similar considerations apply to the phase defect due to capacity

$$\delta_c = \tan^{-1} \frac{\sigma_c}{M\omega} = \frac{\sigma_c}{M\omega} \text{ (approx.).}$$

Equation (3) gives

$$\delta_c = R_1 C_1 \omega + R_2 C_2 \omega - C_{12} \left\{ R_1 \frac{(L_2 - M_0)}{M_0} \omega + R_2 \frac{(L_1 - M_0)\omega}{M_0} \right\}, \quad . \; . \quad (6)$$

and from this it follows that $\sigma_c/M\omega$ is always positive for a negative mutual inductance. In the case of a positive mutual inductance, the phase defect due to mutual capacity opposes that due to self-capacity, when $L_2 > M_0$ and $L_1 > M_0$, and in all cases the impurity and phase defect of a positive inductance are less than those in a negative mutual inductance, if the capacities are the same in the two cases. When we are dealing with one particular pair of coils, the sign of M may be changed by simply changing the common point ; but by reference to Figs. 1 and 2 we see that this change also alters the values of C_1, C_2 and C_{12}, unless the whole arrangement is perfectly symmetrical, which is not probable. Thus, when the sign of M is changed, the frequency coefficient and impurity are liable to change merely on account of the alteration in the values of C_1, C_2 and C_{12}. Nevertheless, in almost all the cases met with in practice, the negative connexion possesses much the larger frequency coefficient and impurity.

BUTTERWORTH* has also shown that the effect of eddy currents in the coils is to introduce another component of impurity σ_e. This may be positive or negative, depending on the method of winding of the coils ; but, in general, standards of mutual inductance are wound with stranded wire, and the effects of eddy currents are of less importance than those due to capacity.

Methods of Measurement.

In setting out to measure the properties of a number of typical standards of mutual inductance, it is important to have a method of measurement of effective mutual inductance and impurity at various frequencies, which is rapid in operation and which

* BUTTERWORTH, *loc. cit.*

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can be used for a wide range of frequencies. The methods used are described in the preceding paper.* Two alternating-current potentiometer methods are there described. By means of one method two mutual inductances of the same sign, and not necessarily of the same magnitude, can be compared for impurity and effective mutual inductance at any desired frequency. The other method, the simplest possible, permits the comparison of two mutual inductances, equal in magnitude but opposite in sign. In both cases the impurity of one mutual inductance is given directly in ohms, provided the impurity of the other is negligibly small, or at any rate known. The quantities which it was desired to investigate were the frequency coefficient, or change of effective mutual inductance with frequency, and the impurity. The procedure was as follows: The instrument under test was connected in a circuit by means of which it could be balanced against a mutual inductance standard, which was known to possess a very small impurity and frequency coefficient. Balance was first obtained at a low frequency—say 100 cycles per second—using a vibration galvanometer as detector. At this low frequency the impurity of any ordinary mutual inductor is sensibly zero, and its effective mutual inductance is practically equal to its geometrical value M_0. The frequency was then raised to, say, 1,000 cycles, and a telephone was substituted for the vibration galvanometer as detector. The change to the higher frequency causes a change in the effective mutual inductance, and also introduces an impurity, and the arrangement is no longer balanced. The instruments tested were all variable mutual inductors, and thus balance at the higher frequency was obtained by reducing the mutual inductance reading by an amount ΔM, and at the same time altering the impurity adjustment r. If the impurity and frequency coefficient of the standard are negligible, then ΔM gives at once the change of mutual inductance with frequency of the instrument under test, whilst the impurity is obtained at once from the change in the value of r. The frequency was then again raised, and the values of ΔM and impurity found for the higher frequency, and so on. In this way the frequency coefficient and impurity of the instrument were readily obtained for the whole range of telephonic frequencies.

In order to work the method successfully, it is necessary to have a high-frequency standard of mutual inductance—that is to say, a standard whose properties at high frequency are as nearly as possible the same as at low frequencies. For this purpose two coils such as are used in radio-frequency work were frequently used. These coils were wound of highly stranded wire, so that eddy current effects were very small, and they were "slice" wound, so that capacity effects were reduced to a minimum. Two such coils of the same diameter and placed side by side possess a mutual inductance which is very nearly pure and invariable with frequency over the telephonic range. The value of the mutual inductance is conveniently varied by moving one coil away from the other along their common axis. They must not, of course, be turned so that a portion of the winding of one coil comes into the strongest part of the field of the other,

* L. HARTSHORN, 'N.P.L. Collected Researches,' Vol. 21, Paper 3, p. 45; or 'J. Scientific Instruments,' Vol. 2, p. 145, 1925.

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otherwise eddy current effects will be increased. For testing 10 mH mutual inductometers, a fixed standard mutual inductance designed by Dr. D. W. DYE was used. The coils are wound on an ebonite frame of cylindrical form. This is " slice " wound with stranded wire (size 7/36's), the whole forming a coil of self-inductance about 50.5 mH. The winding is broken at two points, thus dividing it into three sections (Fig. 4). The middle section serves as the secondary coil, while the other two coils connected in series form the primary. Thus the secondary is sandwiched in between the two halves of the primary, thereby securing a comparatively high coupling coefficient, and thus enabling the 10 mH of mutual inductance to be obtained with correspondingly less wire.

Determination of Residual Effects in the Standard.

For many purposes this standard may be regarded as a perfectly pure mutual inductance of constant value. There will, however, be certain small capacity effects even at telephonic frequencies. In particular, the mutual capacities are likely to be rather

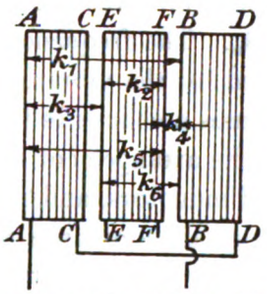

FIG. 4.—Arrangement of Coils in High-Frequency Mutual Inductance Standard.

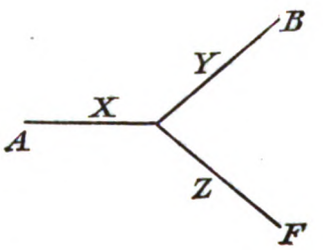

FIG. 5.—Equivalent Star Connexion.

high on account of the closeness of coupling, and these mutual capacities affect the values of C_1, C_2 and C_{12}, as previously explained. Eddy current effects also may not be absolutely negligible. An idea of the magnitude of these residual effects was obtained in the following manner.

(*a*) *Self-capacity Measurements at Radio Frequencies.*—Let one primary terminal be connected to one of the secondary terminals, so that the system of Fig. 2 is obtained. BUTTERWORTH* has shown that this is equivalent to the system of Fig. 5, where X, Y and Z are vector impedances (assuming we are dealing with alternating currents of frequency $\omega/2\pi$), such that the following relations are satisfied,

$$\mathrm{X, Y, Z} = \frac{x + \alpha\mu,\ y + \beta\mu,\ z + \gamma\mu}{1 + \alpha\,(y + z) + \beta\,(z + x) + \gamma\,(x + y) + \lambda\mu},$$

where

$$x = j\omega \mathrm{M}_0 = m, \quad y = \mathrm{R}_1 + j\mathrm{L}_1\omega - j\mathrm{M}_0\omega = l - m, \quad z = \mathrm{R}_2 + j\mathrm{L}_2\omega - j\mathrm{M}_0\omega = n - m,$$

$$\alpha = j\omega \mathrm{C}_{12}, \qquad \beta = j\omega \mathrm{C}_2, \qquad \gamma = j\omega \mathrm{C}_1,$$

$$\mu = yz + zx + xy = ln - m^2, \qquad \lambda = \beta\gamma + \gamma\alpha + \alpha\beta.$$

* BUTTERWORTH, *loc. cit.*

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Let now a variable air condenser be connected across the primary coil (*i.e.*, across AB in Fig. 2), and let the coil be loosely coupled to an oscillating circuit. Let the capacity of the added condenser be varied until the circuit resonates. If this is done for a number of frequencies, the relation between the added capacity and the corresponding resonating frequency can be obtained. The circuit will resonate at the frequency $\omega/2\pi$, such that the reactance of the oscillatory circuit formed is zero. This reactance consists of the reactance component of X + Y, together with that of the added condenser. From the above equations we obtain for the impedance X + Y,

$$X + Y = \frac{l + (\alpha + \beta)(ln - m^2)}{1 + \alpha(l + n - 2m) + \beta n + \gamma l + \lambda\mu}.$$

The capacities C_1, C_2, C_{12} are small, and if the frequency is well below the self-resonance points of the various coils, the terms α, β, γ are small compared with $\frac{1}{l}$, $\frac{1}{m}$, $\frac{1}{n}$, *i.e.*, terms such as $l\alpha$, $n\beta$, $l\gamma$, etc., are small compared with unity. If we neglect squares and products of such small quantities the above expression reduces to

$$\begin{aligned} X + Y &= l - \alpha(l - m)^2 - \beta m^2 - \gamma l^2 \\ &= (R_1 + jL_1\omega) - j\omega C_{12}[R_1 + j(L_1 - M_0)\omega]^2 + j\omega C_2 M_0^2 \omega^2 - j\omega C_{12}[R_1 + jL_1\omega]^2. \end{aligned}$$

The reactance component of this expression is

$$\omega[L_1 - (C_1 + C_{12})R_1^2 + \omega^2\{C_1L_1^2 + C_2M_0^2 + C_{12}(L_1 - M)^2\}].$$

The second term in this expression will be negligibly small for all reasonably good coils, and thus it can be omitted. The reactance of the added condenser is $-1/C\omega$. Thus, the condition for resonance is to a first approximation,

$$\omega L_1\left[1 + \omega^2\left\{C_1L_1 + C_2\frac{M_0^2}{L_1} + C_{12}\frac{(L_1 - M_0)^2}{L_1}\right\}\right] - \frac{1}{C\omega} = 0,$$

or,

$$\begin{aligned} \frac{1}{\omega^2} &= CL_1\left[1 + \omega^2\left\{C_1L_1 + C_2\frac{M_0^2}{L_1} + C_{12}\frac{(L_1 - M_0)^2}{L_1}\right\}\right] \\ &= L_1\left[C + \left\{C_1 + C_2\frac{M_0^2}{L_1^2} + C_{12}\frac{(L_1 - M_0)^2}{L_1^2}\right\}\right] \text{ (approx.)}, \end{aligned}$$

since, in the second term, which is small, we may put $L_1C\omega^2 = 1$ with sufficient approximation. Thus, on plotting corresponding values of C and $1/\omega^2$ we get a straight line. which cuts the axis of C at the point $C = -C_p$, such that

$$C_p = C_1 + C_2\frac{M_0^2}{L_1^2} + C_{12}\frac{(L_1 - M_0)^2}{L_1^2}, \quad \ldots\ldots\ldots \quad (7)$$

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and which is inclined to the C axis at an angle $\tan^{-1} L_1$. Evidently we may regard C_p as the effective self-capacity of the primary coil in the presence of the secondary, and may determine it in the manner indicated. Similarly the effective self-capacity of the secondary in the presence of the primary is given by

$$C_s = C_2 + C_1 \frac{M_0^2}{L_2^2} + C_{12} \frac{(L_2 - M_0)^2}{M_0^2}. \quad \ldots\ldots\ldots \quad (8)$$

and this may be determined in the same way. Finally, by connecting a variable condenser across BF (Fig. 2), and observing resonance points as before, the effective self-capacity of the primary and secondary in series (C_{ps}) may be determined. In this case the impedance of the coils between the terminals BF is given by

$$Y + Z = \frac{(l + n - 2m) + (ln - m^2)(\beta + \gamma)}{1 + \alpha (l + n - 2m) + \beta n + \gamma l + \lambda \mu}.$$

Neglecting small quantities of the second order, this becomes

$$Y + Z = l_s + (\beta + \gamma)(ln - m^2) - \alpha l_s^2 - \beta l_s n - \gamma l_s l,$$

where

$$l_s = y + z = l + n - 2m = R_1 + R_2 + j\omega (L_1 + L_2 - 2M_0).$$

The reactance component of the above impedance is

$$\omega [L_1 + L_2 - 2M_0 - \{C_1 R_1^2 + C_2 R_2^2 + C_{12} (R_1 + R_2)^2\} + \omega^2 \{C_1 (L_1 - M_0)^2 + C_2 (L_2 - M_0)^2 + C_{12} (L_1 + L_2 - 2M_0)^2\}].$$

Again, the term involving the resistance is negligibly small. Let C be the added capacity, and denote $L_1 + L_2 - 2M_0$ by L_s, then the condition of resonance is

$$\frac{1}{\omega_1} = CL_s \left[1 + \omega^2 \left\{ C_1 \frac{(L_1 - M_0)^2}{L_s} + C_2 \frac{(L_2 - M_0)^2}{L_s} + C_{12} L_s \right\}\right].$$

Thus, the self-capacity C_{ps} is given approximately by

$$C_{ps} = C_{12} + C_1 \frac{(L_1 - M_0)^2}{L_s^2} + C_2 \frac{(L_2 - M_0)^2}{L_s^2}. \quad \ldots\ldots\ldots \quad (9)$$

Thus, if L_1, L_2, and M_0 are known, and we measure C_p, C_s, and C_{ps} in this way, we have three equations (7), (8) and (9), connecting C_1, C_{12}, and C_2, from which we can obtain their values.

Measurements were made in this way on the standard mutual inductance. Fig. 4 represents the geometry of this, while Fig. 1 represents its electrical constants. It

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is evident from Fig. 1 that four common points are possible, and by making the appropriate connexions we obtain four different mutual inductances, two positive and two negative (the geometrical value is, of course, the same in each case).

The values observed for the various effective self-capacities are given in Table I. Substituting the values C_p, C_s, C_{ps} for any given common point in equations (7), (8), (9), and solving, the corresponding values of C_1, C_2, C_{12} were obtained. These are also given in Table I. It is interesting to note that the measured effective self-capacities

TABLE I.

Sign of M.	Common point.	Effective Self-capacities, μμF.			Component capacities, μμF.		
		C_p	C_s	C_{ps}	C_1	C_2	C_{12}
+	AE	33	22	70	19 $k_1 + k_6$	6 $k_2 + k_5$	57 k_4
+	BF	16	32	32	5 $k_1 + k_5$	25 $k_2 + k_6$	30 k_3
−	AF	130	170	55	60 $k_1 + k_4$	26 $k_2 + k_3$	30 k_6
−	BE	57	100	26	21 $k_1 + k_3$	60 $k_2 + k_4$	8 k_5

$L_1 = 18{\cdot}5$ mH. $L_2 = 12{\cdot}0$ mH. $M_0 = \pm 10{\cdot}0$ mH.

are much greater when M_0 is negative than when it is positive. This is due to the large change in the numerical value of the terms $(L_1 - M_0)$, etc., in equations (7), (8) and (9). The component capacities C_1, C_2, C_{12} are of the same order in the two cases. Immediately underneath each component capacity in Table I is given the combination of the various end-capacities in Fig. 1, which go to form that particular value of C_1, C_2 or C_{12}. From this it is at once evident that k_4, the capacity between B and F is the greatest of these end-capacities. The reason for this is apparent on inspection of Fig. 4; the points BF are the nearest together. The capacity k_5 between the points A and F is very much smaller, corresponding to their greater distance apart. It is evident from the results in Table I that we cannot regard the capacities k_1, k_2, k_3, etc., as strictly additive when the connexions are such as to throw them in parallel. The capacity C_1 is certainly largest when AF is the common point, in which case the large capacity k_4 is thrown across the primary, but in general, where a capacity may be expected to be equal to the sum of two end-capacities, such as $k_1 + k_6$, the value actually observed is smaller than the sum would be, judging by the values given for the single capacities k_3, k_4, k_5, k_6 as obtained from the values of C_1, C_2, C_{12}. This is partly explained by the fact that the capacities k_1, k_2 represent actions which take place across the same

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dielectric path ; if they were to be strictly additive, each would have to be completely self-contained and shielded from the others.

Using these values of the various capacities, we can now calculate the values of σ and $\Delta M/M_0$ using equations (3) and (5). The values are given in Table II.

TABLE II.—Frequency Coefficient and Impurity of High-frequency Fixed Mutual Inductance Standard. (Values calculated from capacities.)

Sign of M.	Common point.	Frequency, cycles per sec.	$\frac{\Delta M}{M} \times 10^4$.	σ (ohm).	$\frac{\sigma}{M\omega} \times 10^5$.
+	AE	1,000	0·1	−0·0001	−0·2
		2,000	0·5	−0·0005	−0·4
		4,000	2·0	−0·002	−0·8
+	BF	1,000	0·1	$0·000_0$	—
		2,000	0·6	$0·000_0$	—
		4,000	2·2	$-0·000_1$	−0·04
−	AF	1,000	1·3	$-0·001_3$	$+2·_1$
		2,000	$5·_2$	−0·005	+4
		4,000	$21·_0$	−0·021	+8
−	BE	1,000	$0·_5$	$-0·000_5$	$+0·_8$
		2,000	$2·_1$	−0·002	$+1·_6$
		4,000	$8·_3$	−0·008	$+3·_2$

A check on the accuracy of these results was obtained in the following way. The standard was balanced against a variable mutual inductance of the same value, using the simple comparison method, and readings of M and r noted, at a telephonic frequency. The connexions to the primary and secondary coils of the standard were then each reversed, thus changing the common point, but not changing the sign of M. Balance could therefore again be obtained by adjusting the variable mutual inductance and also the impurity adjustment just sufficiently to compensate for the differences in the impurity and frequency change of the standard, for the two conditions of common points, *i.e.*, the method is one of simple substitution, since the frequency errors of the variable mutual inductance are sensibly the same in the two measurements. The results of such measurements are given in Table III, where $\left(\frac{\Delta M}{M_0}\right)_{AF} - \left(\frac{\Delta M}{M_0}\right)_{BE}$ represents the difference of the frequency corrections of the standard when the common points are AF and BE. Within the experimental error, the calculated and observed results are identical. In Table III, the values σ_{AF} and σ_{BE} must be considered as representing the impurity due to capacity effects only, since even in the case of the observed values, although eddy-current effects may also be appreciable, they may be expected to be the same in the two cases, and thus will cancel out on taking differences.

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TABLE III.

Frequency, cycles per sec.	$\left(\frac{\Delta M}{M}\right)_{AF} - \left(\frac{\Delta M}{M}\right)_{BE}$.		$\sigma_{AF} - \sigma_{BE}$.	
	Calculated.	Observed.	Calculated.	Observed.
1,000	$0{\cdot}_8 \times 10^{-4}$	$0{\cdot}_4 \times 10^{-4}$	−0·001	−0·001
2,000	$3{\cdot}_1 \times 10^{-4}$	$2{\cdot}_6 \times 10^{-4}$	−0·003	−0·004
4,000	$10{\cdot}_8 \times 10^{-4}$	$11{\cdot}_8 \times 10^{-4}$	$-0{\cdot}01_1$	$-0{\cdot}0_2$

The importance of eddy-current effects was estimated by the following experiments :—

(*b*) *Effective Resistance Measurements at Radio Frequencies.*—This method was used by BUTTERWORTH. The effective resistances of the coils at radio frequencies were measured by the ordinary resistance substitution method. As before, let the coils have a common point. Let R_1 be the effective resistance of the primary coil AB (Fig. 3) at any given frequency, let R_2 be the effective resistance of the secondary AF at the same frequency, then the resistance of the two coils in series (BAF) at the same frequency is $R_1 + R_2 - 2\sigma$, where σ is the impurity of the mutual inductance between the coils at that frequency. Thus the total σ can be measured at a number of high frequencies, and the value at telephonic frequency obtained by extrapolation. Values obtained for the standard in this way are given in Table IV for two conditions of common point. Under the heading " impurity " column (*a*) gives the results obtained directly from the effective resistance measurements at the frequencies stated, column (*b*) gives

TABLE IV.

Sign of M.	Common point.	Frequency, cycles per sec.	Impurity σ—ohms.		
			Observed. (*a*)	By extrapolation. (*b*)	From capacity measurements. (*c*)
+	AE	37,400	+0·4		
		30,000	+0·2		
		20,000	$+0{\cdot}0_5$		
		4,000		$+0{\cdot}00_1$	−0·002
		2,000		$+0{\cdot}000_2$	$-0{\cdot}000_5$
		1,000		$+0{\cdot}000_0$	$-0{\cdot}000_1$
−	AF	25,000	−1·9		
		20,000	−1·0		
		10,000	−0·2		
		4,000		$-0{\cdot}02_1$	$-0{\cdot}02_1$
		2,000		−0·004	−0·005
		1,000		$-0{\cdot}000_8$	$-0{\cdot}001_3$

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the values obtained for various telephonic frequencies by extrapolation from the results in column (*a*). In column (*c*) the corresponding values of σ_e are given for comparison. These values are taken from Table II. The values in column (*a*) were found to vary as a higher power of the frequency than the square. This is probably due to dielectric losses in the insulation and will be referred to again later. The relation $\sigma = \sigma_0 \omega^{2 \cdot 45}$ was found to hold approximately, and this was used in extrapolating. Even if the square law is assumed, however, the general conclusions are not affected, viz., that the component of σ due to eddy currents is less than 0.01 ohm at 4,000 cycles per sec., and is therefore negligible for all practical purposes at telephonic frequencies. This follows at once from a comparison of the results in columns (*b*) and (*c*), and may be expected with the stranded wire used. Thus the values given in Table II may be taken as completely representing the behaviour of the standard at telephonic frequencies. When used as a positive mutual inductance, the impurity is negligible, while the value of the mutual inductance increases by 2 parts in 10,000 in passing from zero frequency (or, say, 100) to 4,000 cycles per sec. When used as a negative mutual inductance, the impurity just becomes appreciable, whilst in going from 100 to 4,000 cycles per sec., the mutual inductance increases by 2 parts in 1,000 in one case. The importance of knowing this is obvious, since in practice, when testing a positive mutual inductance by the simple comparison method (the most satisfactory), the standard must be used as a negative mutual inductance.

Working Standards of Mutual Inductance.

The Campbell inductometer, or variable mutual inductance standard, is very widely used for alternating-current bridge work, and accordingly it appeared to be advisable to determine its properties in some detail. BUTTERWORTH has given the frequency corrections for a 10 mH instrument, at the top point of its scale, at 1,000 cycles; but, in addition to this, it is important to know how the frequency corrections vary with scale reading, change in common point, and also with the frequency itself. These points were investigated by comparison with the standard mutual inductance described above. This standard consisted of three coils (Fig. 4), and by using various combinations of these coils values of about 2 mH, 5 mH, 7 mH and 10 mH could be obtained. These were compared directly with the Campbell instrument set at the same reading. For the purposes of these experiments, it was usually sufficiently accurate to assume that the standard mutual inductance was perfect. Tables V and VI give the results obtained with the 10 mH Campbell instrument. Table V gives the results for a common point which makes the mutual inductance positive, while Table VI gives values for a common point corresponding to a negative value for M. The following points are to be noted :—

(1) The increase of mutual inductance (ΔM) with increase of frequency is always proportional to the square of the frequency within the experimental error.

(2) When the mutual inductance is positive, the fractional increase of mutual inductance with frequency is approximately constant (at any given frequency) over

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TABLE V.—Frequency Corrections of 10 mH Campbell Inductometer. Common point HA_0. Mutual inductance positive.

Instrument reading M_0. (μH).	Frequency, cycles per sec.	ΔM (μH).	σ. (ohms).	$\frac{\Delta M}{M_0} \times 10^4$.	$\frac{\sigma}{M\omega}$.
1,000	500	$0 \cdot_3$	$+0 \cdot 000_4$	$2 \cdot_9$	$0 \cdot 0001_1$
	1,000	$1 \cdot_2$	$+0 \cdot 001_4$	$12 \cdot_6$	$0 \cdot 0002_2$
	1,500	$2 \cdot_7$	$+0 \cdot 003_2$	26	$0 \cdot 0003_3$
	2,000	$4 \cdot_8$	$+0 \cdot 005_7$	46	$0 \cdot 0004_4$
1,820	500	$0 \cdot_5$	$+0 \cdot 000_7$	$2 \cdot_8$	$0 \cdot 0001_2$
	1,000	$2 \cdot_0$	$+0 \cdot 002_5$	11	$0 \cdot 0002_2$
	1,500	$4 \cdot_5$	$+0 \cdot 006$	25	$0 \cdot 0003_5$
	2,000	$8 \cdot_1$	$+0 \cdot 010$	45	$0 \cdot 0004_4$
5,000	500	$1 \cdot_3$	$+0 \cdot 002_6$	$2 \cdot_6$	$0 \cdot 0001_3$
	1,000	$5 \cdot_4$	$+0 \cdot 007_0$	11	$0 \cdot 0002_2$
	1,500	$12 \cdot_1$	$+0 \cdot 017$	24	$0 \cdot 0003_6$
	2,000	$21 \cdot_7$	$+0 \cdot 033$	44	$0 \cdot 0005_2$
6,820	500	$1 \cdot_7$	$+0 \cdot 002_8$	$2 \cdot_5$	$0 \cdot 0001_3$
	1,000	7	$+0 \cdot 011$	10	$0 \cdot 0002_5$
	1,500	15	$+0 \cdot 026$	22	$0 \cdot 0004_1$
	2,000	28	$+0 \cdot 048$	41	$0 \cdot 0005_6$
10,000	500	$2 \cdot_5$	$+0 \cdot 004$	$2 \cdot_5$	$0 \cdot 0001_3$
	1,000	$9 \cdot_8$	$+0 \cdot 016$	$9 \cdot_8$	$0 \cdot 0002_6$
	1,500	22	$+0 \cdot 040$	22	$0 \cdot 0004_3$
	2,000	39	$+0 \cdot 075$	39	$0 \cdot 0006_0$

TABLE VI.—Frequency Corrections of 10 mH Campbell Inductometer (continued). Common point HC_2. Mutual inductance negative.

Instrument reading (μH).	Frequency, cycles per sec.	ΔM (μH).	σ (ohms).	$\frac{\Delta M}{M_0} \times 10^4$.	$\frac{\sigma}{M\omega}$.
1,820	500	$0 \cdot_5$	$-0 \cdot 000_7$	$2 \cdot_5$	$0 \cdot 0001_2$
	1,000	$2 \cdot_0$	$-0 \cdot 002_6$	11	$0 \cdot 0002_3$
	1,500	$4 \cdot_4$	$-0 \cdot 006_7$	24	$0 \cdot 0003_9$
	2,000	$8 \cdot_4$	$-0 \cdot 013$	46	$0 \cdot 0005_7$
5,000	500	$1 \cdot_2$	$-0 \cdot 002_3$	$2 \cdot_4$	$0 \cdot 0001_4$
	1,000	$5 \cdot_8$	$-0 \cdot 010$	12	$0 \cdot 0003_2$
	1,500	$13 \cdot_0$	$-0 \cdot 026$	26	$0 \cdot 0005_5$
	2,000	$25 \cdot_4$	$-0 \cdot 050$	51	$0 \cdot 0008_0$
6,820	500	2	$-0 \cdot 003_9$	$2 \cdot_9$	$0 \cdot 0001_8$
	1,000	9	$-0 \cdot 017$	13	$0 \cdot 0004_0$
	1,500	20	$-0 \cdot 041$	29	$0 \cdot 0006_3$
	2,000	36	$-0 \cdot 080$	53	$0 \cdot 0009_3$
10,000	500	$3 \cdot_8$	$-0 \cdot 006_0$	$3 \cdot_8$	$0 \cdot 0001_9$
	1,000	15	$-0 \cdot 028$	15	$0 \cdot 0004_5$
	1,500	35	$-0 \cdot 067$	35	$0 \cdot 0007_0$
	2,000	62	$-0 \cdot 134$	62	$0 \cdot 0010_7$

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the whole range of the instrument. This does not hold when the mutual inductance is negative.

(3) The impurity σ in nearly every case increases more rapidly than the square of the frequency. BUTTERWORTH showed that the impurity due to capacity and eddy-current effects is proportional to the square of the frequency under the conditions of these experiments. It is evident that there must be an additional component of impurity, which is proportional to a power of the frequency higher than the square. It is shown in a succeeding paragraph that this may be explained by dielectric losses in the insulating material in the coils.

(4) When the mutual inductance is positive the phase defect of this instrument is approximately constant at any given frequency for all readings of mutual inductance. This is very convenient in practice. For instance, in determining the power factor of a condenser by means of the Carey Foster bridge, using the instrument, the correction to be applied for the impurity of the mutual inductance will always be approximately the same for a given frequency. It is merely necessary to add the phase defect of the instrument to the observed value of the power factor.

(5) When the reading of the instrument is near the top of its scale, the frequency corrections for the negative connexion are much larger than for the positive;

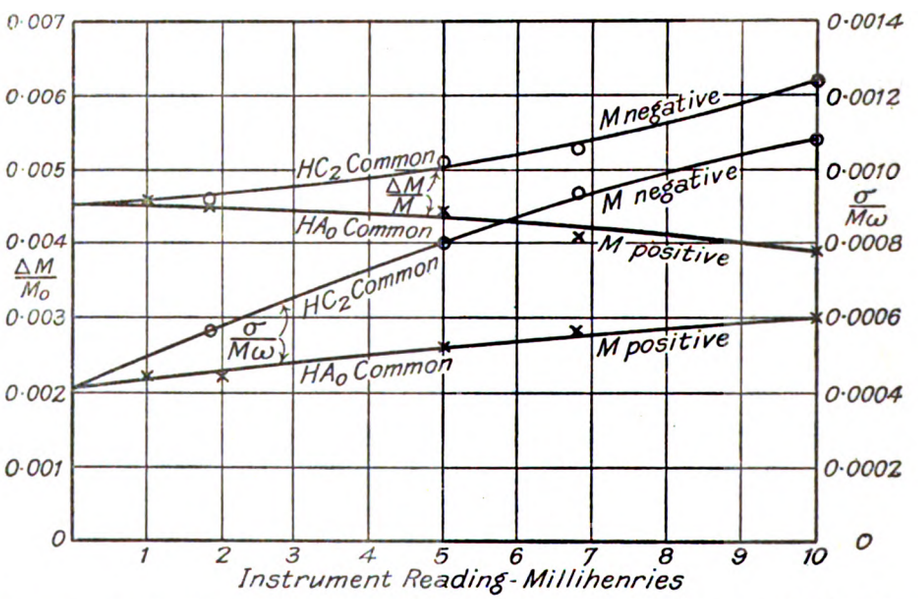

FIG. 6.—Frequency corrections of 10 mH Campbell Inductometer. Frequency = 2,000 cycles per sec.

but as the instrument reading becomes smaller the corrections for the two methods of connexion become more and more nearly equal. Fig. 6 shows this very clearly. There the frequency corrections at a frequency of 2,000 cycles per sec. are plotted

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against instrument readings for both methods of connexion. The differences between the corrections for large values of M are evidently due to the effect of mutual capacity, as previously explained.

Tables V and VI give the results for two common points. Two other common points are possible. The corrections were determined in each case for the reading $M_0 = 10$ mH. The results are collected in Table VII.

TABLE VII.—Frequency Corrections of Campbell Inductometer for Various Common Points. Instrument reading: 10 mH. Frequency: 1,000 cycles per sec.

Common point.	Sign of M.	$\frac{\Delta M}{M_0} \times 10^4$.	σ.	$\frac{\sigma}{M\omega}$.
HA_0	+	$9{\cdot}_8$	$+0{\cdot}016$	$0{\cdot}0002_6$
FC_2	+	$8{\cdot}_2$	$+0{\cdot}015$	$0{\cdot}0002_4$
HC_2	−	$15{\cdot}_8$	$-0{\cdot}028$	$0{\cdot}0004_5$
FA_0	−	$11{\cdot}_8$	$-0{\cdot}021$	$0{\cdot}0003_3$

It is evident that it is important always to use a common point the corrections for which are known.

The values in Tables V, VI and VII all refer to the 10 mH Campbell instrument, which is the most generally useful size of inductometer. The corrections for a 1 mH instrument are somewhat smaller, since the capacity effects are naturally of less importance in coils of smaller self-inductance. Values for a 1 mH instrument at the top of its scale are given in Table VIII. Here again the phase angle error is by no means negligible at 1,000 cycles.

TABLE VIII.—Frequency Corrections of 1 mH Mutual Inductance (Campbell type). Frequency: 1,000 cycles per sec.

Common point.	Sign of M.	$\frac{\Delta M}{M_0} \times 10^4$.	σ.	$\frac{\sigma}{M\omega}$.
HA_0	+	$1{\cdot}_9$	$+0{\cdot}001_5$	$0{\cdot}0002_4$
EC	+	$1{\cdot}_7$	$+0{\cdot}001_5$	$0{\cdot}0002_4$
EA_0	−	$2{\cdot}_4$	$-0{\cdot}001_7$	$0{\cdot}0002_7$
HC	−	$2{\cdot}_5$	$-0{\cdot}001_7$	$0{\cdot}0002_7$

In the case of the 100 mH inductometer of the Campbell type the frequency corrections become much more serious. At 1,000 cycles per sec. the correction on mutual inductance amounts to 1 per cent., and the phase defect is about 0.001. In fact, this instrument is not at all suitable for high-frequency work, and its properties have not therefore

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been examined in such detail. Actual values obtained for a 100 mH Campbell inductometer are given in Table IX. Values of $\Delta M/M_0$ are given for a common point making M positive, and also for one making it negative. When the instrument was used with the middle point of the primary connected to one point on the secondary (a convenient connexion for a self-inductance bridge), the frequency correction was found to be intermediate between those obtained for M positive and M negative respectively. The frequency coefficient $\Delta M/M_0$ was not approximately constant for different M_0 readings in the case of this instrument. For example, at the reading $M_0 = 100$ mH, $\Delta M/M_0$ was about 1 per cent. at 1,000 cycles per sec. At the reading $M_0 = 50$ mH it amounted to 0.5 per cent., while for all readings below 10 mH it was 0.3 per cent. At a given setting, M increased slightly more rapidly than the square of the frequency, but σ increased much more rapidly than the square of the frequency, as is evident from the values given in Table IX.

TABLE IX.—Frequency Corrections of 100 mH Campbell Inductometer.

Common point.	Sign of M.	Frequency, cycles per sec.	ΔM (mH).	σ (ohms).	$\frac{\Delta M}{M_0} \times 10^2$.	$\frac{\sigma}{M\omega}$.
DE	+	500	0·24	$0{\cdot}08_2$	$0{\cdot}2_4$	$0{\cdot}0002_6$
		1,000	1·03	0·60	$1{\cdot}0_3$	$0{\cdot}0009_5$
		1,500	2·41	1·99	$2{\cdot}4_1$	0·0021
		2,000	4·79	4·6	$4{\cdot}7_9$	0·0035
DF	−	1,000	1·42	—	$1{\cdot}4_2$	—

Effect of Dielectric Losses on the Frequency Corrections.

Consider the general case of a mutual inductor with a common point. Let $\xi\eta$ represent the resistance operators of the coils apart from capacity effects and the dielectric losses associated with them. Let their mutual operator be m. The capacity, dielectric loss, and leakage effects may be represented by shunts across the coils, whose conductance operators are α and β respectively, and the corresponding mutual effects (leakage and capacity) may be represented by a conductance (operator γ) across the open ends of the coils. BUTTERWORTH has shown that this system may be reduced by a series of transformations to the system of Fig. 3, in which we have two simple arms (resistance operators ξ' and η') with a mutual operator m', but no shunting conductances. When the terms, α, β, γ are small compared with ξ, η, m, which is the case in all mutual inductance standards at telephonic frequencies, the equivalent operators are given by the approximate equations

$$\left.\begin{aligned} m' &= m + \alpha(\xi - m)(\eta - m) - \beta m\eta - \gamma m\xi \\ \xi' &= \xi - \alpha(\xi - m)^2 - \beta m^2 - \gamma\xi^2 \\ \eta' &= \eta - \alpha(\eta - m)^2 - \beta\eta^2 - \gamma m^2 \end{aligned}\right\}. \quad \ldots\ldots \quad (10)$$

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Considering the case of alternating currents of frequency $\omega/2\pi$ we may replace these operators by vector impedances or conductances. Let R_1, R_2 be the resistances of the coils, L_1, L_2 and M the inductances. The conductance terms α, β, γ must include self-capacity, leakage, and dielectric loss terms. Thus, we may write

$$\left.\begin{aligned} \alpha &= G_1 + j\omega C_1 \quad & \beta &= G_2 + j\omega C_2 \quad & \gamma &= G_3 + j\omega C_3 \\ \xi &= R_1 + j\omega L_1 \quad & \eta &= R_2 + j\omega L_2 \quad & m &= j\omega M \end{aligned}\right\} \quad \ldots\ldots \quad (11)$$

We may also write

$$\left.\begin{aligned} m' &= \sigma + j\omega\,(M + \Delta M) \\ \xi' &= R_1 + \Delta R_1 + j\omega\,(L_1 + \Delta L_1) \\ \eta' &= R_2 + \Delta R_2 + j\omega\,(L_2 + \Delta L_2) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (12)$$

where ΔM, σ, ΔL, etc., may be regarded as the effects of self-capacity, leakage, etc., on resistance, inductance, or impurity. Substituting (11) in (10), and then comparing the result with (12), we find

$$\Delta M = C_{12}R_1R_2 + \omega^2\{L_1C_1M + L_2C_2M - C_{12}(L_1 - M)(L_2 - M)\} - R_1G_1M - R_2G_2M + G_{12}\{R_2(L_1 - M) + R_1(L_2 - M)\}, \quad \ldots \quad (13)$$

$$\sigma = \omega^2[C_1R_1M + C_2R_2M - C_{12}\{R_1(L_2 - M) + R_2(L_1 - M)\}] + G_1L_1M\omega^2 + G_2L_2M\omega^2 - \omega^2G_{12}(L_1 - M)(L_2 - M) + G_{12}R_1R_2, \quad \ldots \quad (14)$$

$$\Delta L_1 = -(C_1 + C_{12})R_1^2 + \omega^2\{C_1L_1^2 + C_2M^2 + C_{12}(L_1 - M)^2\} - 2G_1L_1R_1 - 2G_{12}R_1(L_1 - M), \quad \ldots\ldots \quad (15)$$

$$\Delta R_1 = 2\omega^2R_1\{C_1L_1 + C_{12}(L_1 - M)\} - (G_1 + G_{12})R_1^2 + \omega^2\{G_1L_1^2 + G_2M^2 + G_{12}(L_1 - M)^2\}. \quad \ldots \quad (16)$$

Thus, the frequency coefficient $\Delta M/M$ involves capacity terms of the form $L_1C_1\omega^2$ and also conductance terms of the form G_1R_1. G_1 represents not only leakage such as would be measured by a direct current test, but also the dielectric losses. In a well-constructed coil the D.C. leakage is very small compared with the alternating-current conductance due to dielectric losses. If δ is the loss angle of the insulation then $\tan\delta_1 = G_1/C_1\omega$. G_1 is usually small compared with $C_1\omega$ (*e.g.*, $\tan\delta$ is not likely to exceed 0.05 in a good coil), and R_1 is usually small compared with $L_1\omega$. Thus terms such as G_1R_1 are small compared with those such as $L_1C_1\omega^2$, and, therefore, ΔM will depend almost entirely on the capacity terms. Further, since C_1C_2, etc., are practically independent of the frequency, ΔM will be proportional to the square of the frequency. This fact is verified by the experimental results quoted above.

In the expression for σ, the capacity terms are of the form $C_1R_1M\omega^2$, whilst the conductance terms are of the form $G_1L_1M\omega^2$. When the inductance L_1 is high, this term

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is by no means negligible compared with the capacity term. Also, since G_1 is not even approximately independent of the frequency, this term will not be proportional to the square of the frequency. In a well-constructed coil, the conductances G_1, G_2, etc., will be almost entirely due to dielectric absorption effects in the insulated covering of the wire forming the coils. The conductance due to such effects varies with frequency, at any rate over the telephonic range, in accordance with the formula*

$$G_1 = G_0' \omega^{\alpha},$$

where G_0' is independent of frequency, and α is a constant, for a given material under definite conditions as to temperature, etc. In general, $0 < \alpha < 1$. Thus, the conductance terms in σ are of the form $G'_0 L_1 M \omega^{2+\alpha}$, and consequently σ increases more rapidly than the square of the frequency. This was found to be the case even for a 10 mH inductometer, showing that dielectric loss effects were not negligible even in that case. In the case of the 100 mH inductometer, it was evident that the observed σ was mainly due to conductance arising from dielectric losses, since σ is approximately proportional to $\omega^{2 \cdot 9}$. It is evident that the increase in effective resistance of the primary coil, ΔR_1, will be similarly affected by conductance, while the effect on the change in effective inductance of the primary coil, ΔL_1, will, like that on ΔM, be very small.

The Butterworth-Tinsley Inductometer.

The sub-division of the dials in the Campbell type of inductometer is usually obtained by winding one coil with a cable of 10 strands twisted together, putting the 10 strands in series to form the reading 10 of the dial, and taking tappings from the junctions of the various strands to the studs 1, 2, 3, ..., etc. By this arrangement only one adjustment is necessary for each dial, and the accuracy of sub-division is of a very high order. This method of winding does, however, introduce large capacity effects into the coil in question, and the frequency errors are correspondingly large at high frequencies. BUTTERWORTH, suggested that, for work at high frequencies, a more perfect mutual inductance might be obtained by using for each dial three mutual inductances with a common primary, the values being the 6, 3 and 1 of the dial in question. These three coils must be independently adjusted, but the other points on the dial are obtained by addition and subtraction of the three, *e.g.*, 10 is given by $6 + 3 + 1$, 9 by $6 + 3$, 8 by $6 + 3 - 1$, and so on. It is, of course, necessary to devise a form of switch for connecting up the coils in the proper manner. The coils not required at any particular setting of the dial are left isolated—*e.g.*, at the setting 9, the 6 and 3 coils are connected in series, and the 1 coil is disconnected.

The capacity effects in coils of this construction are much smaller than in the stranded cable coils, and thus the frequency corrections are correspondingly smaller. An inducto-

* See, for example, MCLEOD, 'Phys. Rev.,' Vol. 21, p. 53, 1923; GRANIER, 'Bull. Soc. Française Elect.,' Vol. 3, p. 333, 1923.

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meter constructed in this manner by the firm of Messrs. H. Tinsley & Co. was tested in the same way as the Campbell instrument previously referred to. The values obtained are given in Table X. The frequency corrections are seen to be of the order of one-third to one-seventh of the corresponding corrections in the Campbell instrument, and the new instrument has, therefore, considerable advantages for high-frequency work. The Campbell instrument has, however, important advantages for standardising work which can be carried out at low frequency. It was, of course, designed for such work, since the accuracy of sub-division of the dials is usually greater, and the form of dial used readily lends itself to the calibration of each reading in terms of the maximum reading, without the aid of another standard.

TABLE X.—Frequency Corrections of 10 mH Butterworth-Tinsley Inductometer.

Common point.	M reading (μH).	Frequency, cycles per sec.	ΔM (μH).	σ (ohms).	$\frac{\Delta M}{M} \times 10^4$.	$\frac{\sigma}{M\omega} \times 10^4$.
DC	+5,000	1,000	0·7	$0{\cdot}001_5$	$1{\cdot}_3$	$0{\cdot}_5$
		2,000	$2{\cdot}_6$	0·007	$5{\cdot}_2$	$1{\cdot}_1$
	+10,000	1,000	$1{\cdot}_3$	0·004	$1{\cdot}_3$	$0{\cdot}_6$
		2,000	$5{\cdot}_2$	0·015	$5{\cdot}_2$	$1{\cdot}_2$
DB	−10,000	1,000	$-3{\cdot}_1$	$-0{\cdot}008_7$	$3{\cdot}_1$	$1{\cdot}_3$
		2,000	−12	$-0{\cdot}004_4$	12	$3{\cdot}_5$
		4,000	−51	−0·21	51	$8{\cdot}_4$

It is evident that dielectric losses are also appreciable in this instrument, since σ (when DB is common) increases more rapidly than the square of the frequency.

In conclusion, I wish to express my thanks to Messrs. D. W. MARTIN and D. A. OLIVER of the Electricity Department for assistance in taking some of the observations recorded in this paper.

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V.

NOTE ON SHIELDED NON-INDUCTIVE RESISTANCES

BY

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ALSO

A GENERAL THEOREM ON SCREENED IMPEDANCES

BY

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ABSTRACTS.

Note on Shielded Non-Inductive Resistances.

The residual inductance of a " non-inductive " resistance coil of high value (10,000 ohms or more) changes appreciably with changes of its capacity to earth, and thus it is necessary to shield such coils in precision work. The coil thus becomes a three-terminal device, and it is necessary to define exactly what is meant by its impedance. It is shown that in practice there are three possible values for the impedance of any coil, which are important in bridge and many other measurements. The effect of capacity to screen on these values is pointed out, and a simple equivalent network for the coil is indicated.

A General Theorem on Screened Impedances.

For accurate measurements, especially at high frequencies, it is necessary that all impedances and apparatus used should be surrounded by metal screens kept at definite potentials. By this means, the leakage and capacity effects to earth can be rendered perfectly definite, and the apparatus electrically independent of the surrounding objects. Even when this is done, there is some ambiguity regarding the value of the impedance of the apparatus. The impedance is the ratio of the potential difference across the apparatus to the current. Now both the potential difference and the current depend on the potential of the screen; moreover, the current may be measured at various points. One of the most usual values of the impedance is obtained when the screen is kept at the same potential as the terminal where the current is being measured, but is not directly connected to that terminal. On this definition of impedance it would be natural to expect that there would be two possible values, depending on which terminal is kept at the potential of the screen. In this paper the curious result is obtained that these two values are equal, and certain limitations are considered. This general result is of considerable importance, as it makes it unnecessary to connect the apparatus in any special way, so long as the screen is kept at the same potential as the terminal at which the current is measured.

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NOTE ON SHIELDED NON-INDUCTIVE RESISTANCES.

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Non-inductive resistances of the order of several thousand ohms are becoming common laboratory standards owing to the increasing importance of measurements at telephonic frequencies and the increasing accuracy required at commercial frequencies. It is important in many cases to know the value of the phase angle of these so-called non-inductive resistances. In the case of high resistances, say of several thousand ohms, the effect of capacities becomes an important and often dominating factor of the quadrature component of the pressure across the resistance, and care must be taken that these capacities are independent of neighbouring objects. This necessitates surrounding the resistance with a conducting screen maintained at some definite potential, and it becomes necessary to consider the effect of capacities to the screen on the residual inductance.

We must now define exactly what is meant by the impedance of a screened resistance standard. Consider the resistance AB (Fig. 1), which we will assume completely surrounded by the screen S. This screen is connected to a source of constant potential,

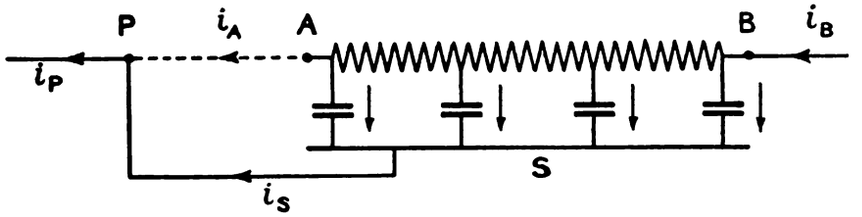

Fig. 1.

generally to earth, and we shall confine our attention to this case. The impedance of any element of a circuit is defined as the ratio of the voltage across the element to the current through it. The current through the resistance AB for a given potential difference across it cannot, however, be expressed as a single definite value, since, owing to distributed capacity effects, the current is not the same in every portion of the coil. In practice we are only concerned with the current flowing in at one terminal, say A, and that flowing out of the other, B, but it is easy to see that these currents are not functions merely of the potential difference between A and B. In general, they depend also on the absolute potentials V_A or V_B, since the capacity currents flowing to the screen will vary as this potential varies, even if the potential difference $V_A - V_B$ remains

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constant. Thus it is only possible to assign a definite value to the impedance of AB by adopting some arbitrary definition of impedance which imposes a definite value on V_A or V_B. We shall assume as our condition that the point A is kept at earth potential (though not necessarily directly connected to earth), as this is the condition most frequently used in bridge measurements. The point P is some other point of the circuit, which is directly connected to earth, so that there is no difference of potential between P and A. Now there is a capacity between each element of the resistance coil and the screen. In the diagram this is represented by a few condensers. Let i_B be the current flowing in at B. Capacity currents will flow from each element of the resistance to the screen. The sum of these currents i_S will return to the circuit at P, so that we have

$$i_B = i_P = i_A + i_S.$$

Now defining* the impedance of AB as the potential difference, V, between A and B, divided by the current at A or B, we obtain two values for the impedance given by

$$Z_{AO} = \frac{V}{i_A}$$

and

$$Z_{AS} = \frac{V}{i_B} = \frac{V}{i_A + i_S}.$$

If now B is kept at earth potential instead of A, the current distribution will be different unless perfect symmetry exists in the construction of resistance coil and screen, which is unlikely. Thus two other values for the impedance are obtained,

$$Z_{BO} = \frac{V}{i_B'},$$

and

$$Z_{BS} = \frac{V}{i_B' + i_S'}.$$

It will be shown, however, that Z_{AO} and Z_{BO} are identical, and therefore each may be represented by Z_O, so that, as long as one terminal is kept at earth potential, there can only be three values of the impedance, namely the two values Z_{AS}, Z_{BS} when the current considered includes the earth current i_S, and one value Z_O when the earth current is not included.

In practice this reduces to :

(1) The impedance is Z_{AS} when A is directly connected to the screen.
(2) The impedance is Z_{BS} when B is directly connected to the screen.
(3) The impedance is Z_O when either A or B is kept at earth potential, but is not directly connected to earth.

* A somewhat similar definition is used by SILSBEE, ' Bureau of Standards, Sci. Papers,' Vol. 20, Paper 516, p. 489, 1926, and also by R. M. WILMOTTE, ' Phil. Mag.,' Vol. 2, p. 65, 1926.

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To prove that $Z_{AO} = Z_{BO}$, consider the simple case of Fig. 2. It is here assumed that there is only one capacity C to earth, the impedances on each side of C being represented by $R_a L_a C_a$ and $R_b L_b C_b$. The effective impedance Z_a between A and C can be shown to be, if we neglect products and higher powers of L_a and C_a,

$$Z_a = R_a + (L_a - R_a^2 C_a)\alpha, \quad \ldots\ldots\ldots\ldots \quad (1)$$

where $\alpha = 2\pi$ (frequency) $\times j$.

Similarly

$$Z_b = R_b + (L_b - R_b^2 C_b)\alpha. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Now

$$i_S = i_A Z_a C\alpha,$$

therefore

$$i_B = i_A (1 + Z_a C\alpha).$$

But

$$V = Z_a i_A + Z_b i_B$$

$$= i_A (Z_a + Z_b + Z_a Z_b C\alpha),$$

therefore

$$Z_{AO} = Z_a + Z_b + Z_a Z_b C\alpha. \quad \ldots\ldots\ldots\ldots \quad (3)$$

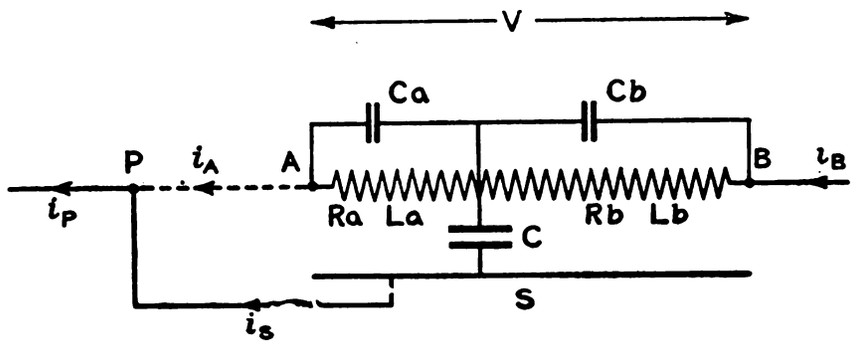

FIG. 2.

It will be noticed that no approximation has been made in obtaining equation (3). Since this equation is symmetrical with regard to Z_a and Z_b it is obvious that $Z_{AO} = Z_{BO}$.

It should be noticed, incidentally, that the symmetry of equation (3) is absolutely independent of the values of Z_a and Z_b. These might be resistances, capacities, self and mutual inductances, or combinations of these; the result $Z_{AO} = Z_{BO}$ would still hold.

The process can be made more general by inserting an unlimited number of condensers such as C, when it can be shown that the equation for Z_0 is symmetrical so long as either A or B are kept at earth potential.

This result was tested experimentally with a 10,000-ohm resistance. The residual inductance was measured on a bridge with a Wagner earth, so that the terminals were not directly connected to earth. The resistance was completely shielded. The residual

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inductance was measured firstly with terminal A, then with terminal B, at earth potential. The arrangement was then made unsymmetrical by putting a condenser of 50 $\mu\mu$ F. between the screen and a tapping point dividing the resistance in the ratio 1 : 9, and the measurements repeated. The values are given in Table I.

TABLE I.

Terminal at earth potential.	Condition.	Residual inductance, mH.
A	Without condenser	$0{\cdot}5_6$
B	,,	$0{\cdot}5_3$
A	With condenser	$0{\cdot}9_5$
B	,,	$0{\cdot}9_9$

It will be noticed that the difference in the measured inductance obtained by keeping first A, then B, at earth potential is very small and about equal in both cases. This difference is considered to be due to the change in the position of the leads necessitated by the reversal of the resistance.

By means of equations (1), (2) and (3), we obtain as a first approximation when the terms L_a, L_b, C_a, C_b, are small

$$Z_0 = R_a + R_b + (L_a + L_b - R_a^2C_a - R_b^2C_b + R_aR_bC)\,\alpha. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From this it is seen that the terms containing C_a and C_b are negative, while that containing C is positive. This is a very useful property, for by judicious use of small external condensers the residual inductance can be made zero. If a resistance is found when constructed to have a positive residual inductance, this can be balanced by a condenser across it, while if the value of the residual inductance is negative, a condenser from some point (not at the ends) of the resistance to the screen can make its value zero.

These remarks only apply to Z_0; the imaginary part of both Z_{AS} and Z_{BS} is always rendered more negative, wherever condensers are added.

We shall now see how these considerations affect bridge measurements. Consider the four-armed bridge in Fig. 3. We shall suppose that Z_{AB} is a non-inductive resistance and whatever conclusions apply to it will also apply to other non-inductive resistances in the other arms of the bridge.

When the bridge is balanced, we have

$$Z_{AB} \,.\, Z_{CD} = Z_{BC} \,.\, Z_{DA}.$$

This holds when the impedances are defined in relation to the current at the telephone end, for this current will be common to two arms of the bridge.

Two cases are of practical importance.

The first occurs when one corner of the bridge, say A, is directly connected to earth. The shield of AB is also earth-connected and is thus directly connected to A, and the impedance will be Z_{AS} as defined above, since the current at B is the same as that at A (Fig. 3) and includes the current flowing to the screen.

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The second case occurs when a Wagner earth point is used. By this means the point B is kept at earth potential without being directly connected to earth. The screen of AB is directly connected to earth, so that the current at B does not include the current to screen, and the effective impedance of the arm AB is now Z_0.

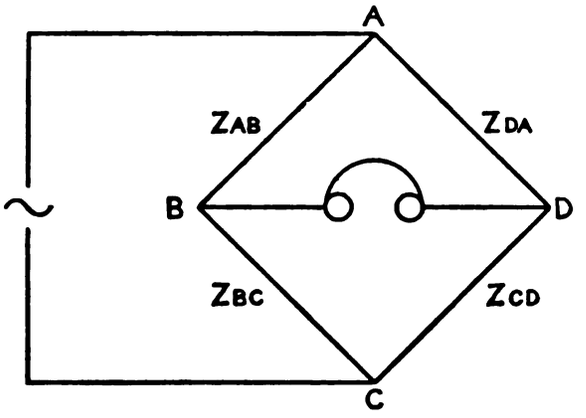

FIG. 3.

It is to be noted that by using such a bridge the three values of the impedance of the shielded resistor AB may readily be determined. In each case the bridge measures the impedance with reference to the current at B.

(1) When the screen is directly earthed, Z_0 is measured as explained above. Reversing the resistance in the bridge does not change its effective impedance.

(2) If the screen is connected to A (and not earthed), the current at B includes i_s, and thus Z_{AS} is determined. The capacity current from the screen to earth does not reach B and thus does not affect the result. Reversing the resistance (with the screen still connected to A) does not affect the measured impedance, since the current measured still includes the capacity current between coil and shield, and there is no capacity current between the screen and earth, since the screen is now at earth potential.

(3) By connecting the shield to B instead of to A the impedance Z_{BS} is determined in exactly the same way.

It is very important to take particular care regarding the values of impedance which should be used in any special case.

Two typical shielded inductionless resistances have been measured, and the values given in Table II show the order of differences which are to be expected in the case of well-made resistances.

TABLE II.

Resistance.	Condition of use.	Effective self-inductance, mH.
10,000 ohms X	Z_0	$-0{\cdot}3_3$
	Z_{AS}	$-0{\cdot}9_4$
	Z_{BS}	$-1{\cdot}0_7$
10,000 ohms Y	Z_0	$+0{\cdot}5_6$
	Z_{AS}	$-1{\cdot}2_7$
	Z_{BS}	$-3{\cdot}1_6$

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The measurements were made on a simple unequal arm bridge in the manner described above. The values were obtained in terms of a standard 1,000-ohm resistance, the effective inductance of which had been determined in terms of a calculable standard.

Since a resistance standard used in this manner possesses three definite values of impedance and only three, it follows that it may be completely represented by the simple network of Fig. 4, where to a first approximation

$$Z_0 = R_{AB} + jwL_{AB},$$
$$Z_{AS} = R_{AB} + jw\,(L_{AB} - R^2_{AB} \cdot C_{BS}),$$
$$Z_{BS} = R_{AB} + jw\,(L_{AB} - R^2_{AB} \cdot C_{AS}).$$

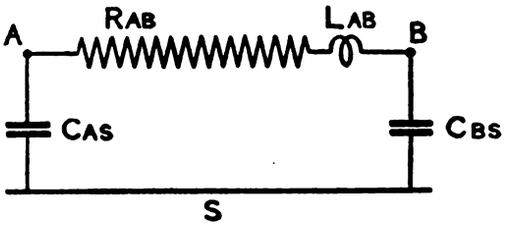

FIG. 4.

In some respects it is more convenient to deal with the three constants L_{AB}, C_{AS}, C_{BS} than it is to consider the three possible values of the impedance. The values of these constants for the two coils X and Y (Table II) are given in Table III.

TABLE III.

Coil.	L_{AB}. mH.	C_{AS}. $\mu\mu$F.	C_{BS}. $\mu\mu$F.
X	$-0{\cdot}3_3$	$7._4$	$6._1$
Y	$+0{\cdot}5_6$	$36._6$	$17._7$

A GENERAL THEOREM ON SCREENED IMPEDANCES. By R. M. WILMOTTE, M.A., Sometime Assistant in the Electricity Department.

[Reprinted from the 'Philosophical Magazine,' Vol. 6, 1928.]

Introduction.

Capacity effects to earth have brought about the universal use of metal screens surrounding any apparatus which is to be used for accurate measurements, especially at high frequencies. The only case in which it is legitimate to do without metal screens is when the impedance of the apparatus is small.

Generally, then, an impedance has three terminals—one at each end, and one on the screen. Let these be A, B and O respectively, as shown in Fig. 1.

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The value of the impedance is the ratio of the potential difference between A and B to the current. This will depend on the potential at which the shield O is kept relative to A or B and the point at which the current is measured. The screen S is usually

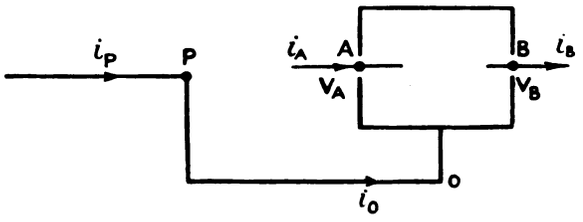

FIG. 1.

connected to some point P in the circuit. This point P is not necessarily directly connected to A or B, but is instrumental in fixing the potential of O relative to A or B. For any setting of the potential of the screen O relative to A, there will be two possible values of the impedance, namely,

$$\frac{V_A - V_B}{i_A} \quad \text{and} \quad \frac{V_A - V_B}{i_B}.$$

If the screen O is given the same setting of potential, but relative to B instead of to A, there should be another two values for the impedance:

$$\frac{V_A' - V_B'}{i_A'} \quad \text{and} \quad \frac{V_A' - V_B'}{i_B'}.$$

It is proved in the next paragraph that the only setting of the potential of O which will make two of these values equal in the general case is when the potential of O is made equal to that of A or B, then

$$\frac{V_A - V_B}{i_A} = \frac{V_A' - V_B'}{i_B'}. \qquad \ldots\ldots\ldots\ldots \quad (1)$$

For any special case, of course, some of the values of the impedance can be made equal by suitable settings of the potential of the screen; for instance, if there is absolute symmetry about the centre of the impedance, it will also be found that

$$\frac{V_A - V_B}{i_B} = \frac{V_A' - V_B'}{i_A'}.$$

In this particular case, then, the values of the impedance can be reduced to two.

The importance of the equality (1) in the general case is that the impedance of any piece of apparatus can be given without stating which terminal should be kept at the potential of the screen. The potential of the screen must be kept at the potential of one of the terminals A or B, and the screen must not be directly connected to that

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terminal. This latter condition is necessary, for if the point P coincided with A, the current measured would not be I_A, but I_P.

The condition that the screen should be kept at the potential of one of the terminals while there is no direct connexion between the two, happens to be a very frequent condition of bridge measurements where a Wagner earth is used. It would be convenient, therefore, to standardise this particular value of the impedance, so that only when other values were required should it be necessary to state the conditions of measurement.

Two other values sometimes required occur when the screen is directly connected to one of the terminals. That is, P coincides with either A or B. If P coincides with A, the current I_B should be measured and not I_P. The reason for this is that, owing to the comparatively large size of the screen, appreciable current may flow *viâ* the earth to the screen from other apparatus in the circuit, and the current I_P will depend not only on the potential difference between A and B, but also on the circuit arrangement. On the other hand, the current i_B is independent of the circuit, being equal to $(i_A + i_0)$.

The three common values of an impedance can be represented diagrammatically by a network of three impedances, as shown in Fig. 2. This method of representation may be useful in some cases (*e.g.*, a condenser) in order to gain a physical conception why the

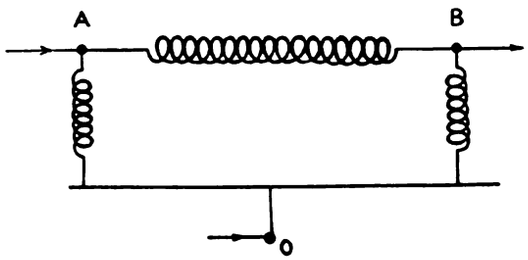

FIG. 2.

different values exist; but in general this is unnecessary, since in any case the complete data of the impedance of a piece of apparatus used in such a way that the screen is kept by some means or other at the potential of one of the terminals, must consist of at least three values.

The above results have been stated for a special case in the preceding paper,* but no proof was given and the result was not generalised, nor were the limitations considered.

Proof.

Consider the impedance made of a network of admittances interconnecting the points 1, 2, 3, ..., n, such that Y_{rs} is the admittance joining the points r and s. The

* See p. 79.

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points 1 and n correspond to the terminals of the apparatus. Let a current I_1 enter at the point 1 and I_n leave at the point n. Let $V_1, V_2, V_3, \ldots, V_n$ be the potentials of the junction points of the admittances. Let the screen be denoted by the point O.

Since the total current entering a junction point is zero, we have

$$I_1 = (V_0 - V_1)\,Y_{01} + (V_2 - V_1)\,Y_{12} + (V_3 - V_1)\,Y_{13} + \ldots + (V_n - V_1)\,Y_{1n}$$

$$= -\,V_1 \sum_{r=0}^{r=n} (Y_{1r}) + V_2 Y_{12} + V_3 Y_{13} + \ldots + V_n Y_{1n} + V_0 Y_{01}.$$

Similarly,

$$0 = V_1 Y_{12} - V_2 \sum_{0}^{n} (Y_{2r}) + V_3 Y_{23} + \ldots + V_n Y_{2n} + V_0 Y_{02}$$

. . . . . .

$$-I_n = V_1 Y_{1n} + V_2 Y_{2n} + V_3 Y_{3n} + \ldots - V_n \sum_{0}^{n} (Y_{rn}) + V_0 Y_{0n}.$$

Eliminating $V_2, V_3, \ldots, V_{n-1}$ from the first $(n-1)$ equations, we obtain

$$I_1 \nabla = V_1 \nabla_1 + V_n \nabla_n + V_0 \nabla_0, \qquad \ldots\ldots\ldots \quad (2)$$

where

$$\nabla = \begin{vmatrix} -\sum_{0}^{n} (Y_{2r}) & Y_{23} & \ldots\ldots & Y_{2\cdot n-1} \\ Y_{23} & -\sum_{0}^{n} (Y_{3r}) & \ldots\ldots & Y_{3\cdot n-1} \\ \ldots & \ldots & \ldots & \ldots \\ Y_{2\cdot n-1} & Y_{3\cdot n-1} & \ldots\ldots & -\sum_{0}^{n} (Y_{r\cdot n-1}) \end{vmatrix}$$

and, unless $r = 1$,

$$\nabla_r = \begin{vmatrix} Y_{1r} & Y_{12} & Y_{13} & \ldots\ldots & Y_{1\ n-1} \\ Y_{2r} & -\sum_{0}^{n} (Y_{2r}) & Y_{23} & \ldots\ldots & Y_{2\cdot n-1} \\ Y_{3r} & Y_{23} & -\sum_{0}^{n} (Y_{3r}) & \ldots\ldots & Y_{3\cdot n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ Y_{n-1\cdot r} & Y_{2\cdot n-1} & Y_{3\cdot n-1} & \ldots\ldots & -\sum_{0}^{n} (Y_{r\cdot n-1}) \end{vmatrix}$$

If $r = 1$, Y_{1r} becomes $-\sum_{0}^{n} (Y_{1r})$ in the determinant for ∇_r,

It should be noted that both ∇ and ∇_n are symmetrical with respect to 1 and n, *i.e.*, 1 and n can be interchanged without changing their value. This symmetry does not exist in ∇_0 or ∇_1. The following relation can also be shown to obtain

$$\nabla_0 + \nabla_1 + \nabla_n = 0. \qquad \ldots\ldots\ldots \quad (3)$$

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Now let V_0, the potential of the screen, be given a certain value between V_1 and V_n, so that it can be written in the form

$$V_0 = V_1 + k(V_n - V_1), \qquad \ldots\ldots\ldots\ldots \quad (4)$$

where k is a constant.

Equations (2), (3), and (4) reduce to

$$\frac{I_1}{V_n - V_1} = \frac{\nabla_n + k \cdot \nabla_0}{\nabla}. \qquad \ldots\ldots\ldots\ldots \quad (5)$$

The left-hand side of this equation is the admittance of the apparatus with the screen at a potential $k(V_n - V_1)$ above that of the point 1, and the current measured at the point 1. This value of the admittance will only be equal to the value when the screen is at a potential $k(V_n - V_1)$ above the point n and the current measured at n, if the right-hand side of equation (5) is symmetrical with respect to 1 and n. This will only occur for either of the two conditions:

(a) that ∇_0 be symmetrical with respect to 1 and n, or

(b) that $k = 0$, that is, $V_0 = V_1$.

The first of these conditions will only occur in special cases, while the second only means that the potential of the screen must be kept at the potential of one of the terminals. The admittance in this latter case is

$$\frac{\nabla_n}{\nabla}.$$

The effect on the admittance of measuring the current at the other terminal is equivalent to putting $k = 1$ in equation (5). The admittance is then

$$\frac{\nabla_n - \nabla_0}{\nabla}.$$

This is not symmetrical with respect to 1 and n in the general case unless ∇_0 is symmetrical. There will, therefore, generally be two values for this admittance, depending on which terminal is kept at the potential of the screen.

It may be noted that in the proof no restriction whatever was imposed on the nature or magnitude of the admittances, so that the result will hold even if the admittances to the screen are not capacitative. Nor were there any limitations to the number of admittances. The existence of distributed admittances, therefore, will not invalidate the result.

Limitations,

In the proof given in the last paragraph, it has been tacitly assumed that the values of the admittances are the same whether $V_0 = V_1$ or V_n, although, admittedly, the current distribution will have altered. In other words, the potential difference between any two consecutive junction points must be proportional to the current flowing directly between those two points. It would appear, therefore, that mutual inductances within the apparatus would invalidate the theorem. That this is so in a general case can be

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proved by working out a simple network. There is, however, a condition in which the presence of mutual inductances does not affect the result, and these cases are so general that they will almost invariably be found in practice.

A mutual inductance, as in Fig. 3 (*a*), between two coils *pq* and *rs*, which are joined at the point *t* through impedances Z_1 and Z_2, can be exactly represented by the network of impedances shown in Fig. 3 (*b*), so long as the current in Z_1 is the same as that in L_1 and the current in Z_2 is the same as that in L_2. Such a mutual inductance, if present in an apparatus, can be replaced by its equivalent network, so that the theorem will still hold, as the assumption on which it depends is accurately correct.

In practice it often happens that the currents in Z_1 and L_1, or Z_2 and L_2, are not absolutely equal owing to small stray capacity effects. These will cause a deviation

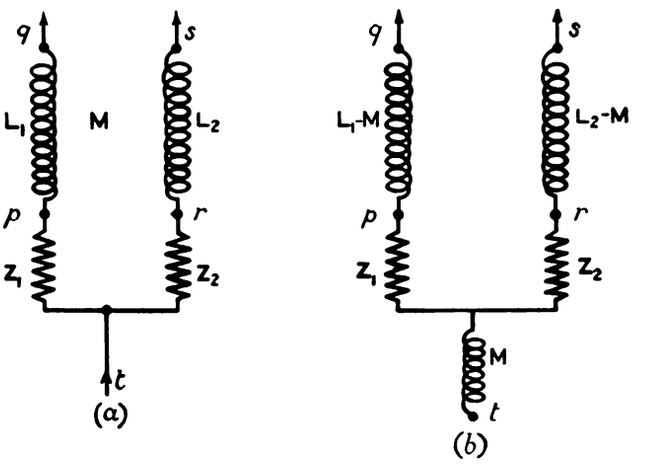

FIG. 3.

to the first order of small quantities. If, moreover, the mutual inductance is also small, the error will depend on the product of the capacity and the mutual inductance, and will only be of the second order.

Another effect that may invalidate the theorem is the presence of eddy currents. This is purely due to a mutual inductance between a coil and a lump of metal, and can be treated as an ordinary mutual inductance. The theorem will still be correct, therefore, if the coil is connected to the lump of metal through an impedance carrying the same current as the coil.

The final cause owing to which the theorem will fail is closely connected with the exact definition of potential difference. The lines of electric intensity are not solely due to the distribution of charges or current alone, but to a combination of the two. Any alteration of one which is not followed by a similar alteration in the other will affect the lines of flow of the displacement current in the dielectric, and will cause an alteration in the distribution of the capacities. In a general case, the distribution of capacities will differ with the potential of the screen. In all practical cases, however, this effect will be quite negligible, with certain exceptions, when the size of the apparatus is comparable to the wave-length.

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VI.

ON THE CALCULATION AND APPLICATION OF HIGH RESISTANCES OF SMALL SELF-INDUCTANCE FOR ALL FREQUENCIES

BY

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ABSTRACT.

In many accurate measurements it is necessary to know the exact phase angle of high resistances of the order of several thousand ohms. In this case the capacity effects are often very large. In this paper, formulæ are given for the self-inductance of high resistances formed by long parallel wires of circular cross-section and screened from earth by a cylindrical metal shield.

The formulæ for the use of such a standard as a ratio arm and as a potential divider are also given, together with its application in bridges.

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ON THE CALCULATION AND APPLICATION OF HIGH RESISTANCES OF SMALL SELF-INDUCTANCE FOR ALL FREQUENCIES.

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I. Introduction.

The use of high resistances in alternating-current measurements often leads to difficulties, owing to the residual self-inductance and the effect of capacity on the circuit. These effects become more pronounced as the frequency increases, and the effect of capacity, which is quite negligible for low resistances, usually becomes predominant when the resistances have values of several thousand ohms.

It has been usual to take as a standard of comparison of residual inductance a single long loop of wire with parallel sides, the formula for which was given by S. L. Brown.* It will be shown, however, that this formula only holds in practice if the middle point of the loop is at earth potential—a condition which does not usually obtain.

For high resistances a single loop of wire forms a very cumbersome standard, and in this paper formulæ are obtained for a standard made of a large number of long parallel wires connected as shown in Fig. 1, where the wires are not necessarily in the same plane.

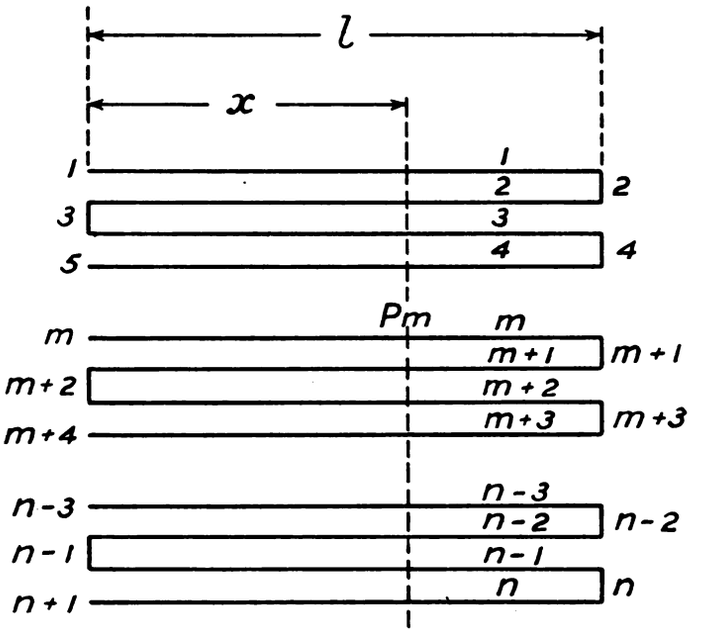

Fig. 1.

* S. L. Brown, 'Phys. Rev.,' Vol. 29, p. 369, 1909.

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It is necessary, especially for high resistances, to screen the wires from the earth by means of a cylindrical metal shield.

Unfortunately, in order to obtain a result which may be computed without too much labour, an approximation is necessary, which entails locating the wires in fairly close proximity to the screen. This is not very satisfactory, because, though the residual inductance may be calculated, it will generally have a comparatively large negative value. When the number of wires is not greater than four, however, the necessity for this approximation vanishes, and the formulæ derived for such cases should be useful in designing resistances, particularly for high-frequency work.

The adaptation to high voltages is complicated, since a very considerable length of wire is required, owing to the necessity of minimising the temperature rise. For such purposes voltages of the order of 2 to 10 per metre length of wire are commonly used, the current being 0.1 ampere or less. There is also the limitation imposed by the formation of corona around the wire, due to the electric field between it and the shield. These considerations would require a number of systems of the type suggested to be put in series for high voltages. A phase displacement would result, depending upon the capacity and voltage of the different shields to earth. This effect can, however, be calculated if the capacities of the shields to earth are known (either by measurement, or by surrounding the whole with a large concentric screen connected to earth and calculating the capacities). In this connexion, when dealing with commercial frequencies, non-inductive resistances for high voltages can be constructed more satisfactorily by the method described by E. Orlich and H. Schultze,* which is more compact and allows of easier computation, but is not suitable for higher frequencies.

Formulæ are also given for the use of the standard as ratio arms and as potential divider, and it is hoped that they may be found useful at audio frequencies and in developing bridge methods suitable for radio frequencies.

II. Capacity Coefficients of n thin parallel Wires within a Cylinder.

The problem of the capacity between infinite parallel cylinders has not been completely solved, but an approximate solution can be found if the radii of the cylinders are small compared with their distances apart.

Consider n parallel thin conductors of radii $r_1, r_2, \ldots r_n$, having polar co-ordinates $(a_1, \theta_1), (a_2, \theta_2), \ldots (a_n, \theta_n)$, referred to the centre of the cylindrical metal screen of radius R as origin. The image of the wire A_r in the screen will be B_r, where $OB = R^2/a_r$.

Let $q_1, q_2, \ldots q_n$ be the charges on the wires, the potentials of which are $v_1, v_2, \ldots v_n$ respectively. We shall assume the dielectric constant to be unity.

Then, the potential of the screen will be given by

$$v_s = C - 2\sum_{r=1}^{r=n} q_r \log (a_r/R), \quad \ldots\ldots\ldots\ldots (1)$$

* Orlich and Schultze, 'Arch. Elektrotech.,' Vol. 1, p. 1, 1912.

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and the potential at the surface of the wire of radius r_m surrounding A_m is, if r_m is small, approximately given by

$$v_m = C + 2q_m \log\left(\frac{R^2 - a_m^2}{a_m r_m}\right) + 2\sum_{r=1}^{r=n} q_r \log\left(\frac{A_m B_r}{A_m A_r}\right), \quad \ldots\ldots\ldots (2)$$

in which the term having $r = m$ is left out in the last sum.

If the screen is connected to earth, we have

$$v_m = 2q_m \log\left(\frac{R^2 - a_m^2}{R r_m}\right) + 2\sum_{r=1}^{r=n} q_r \log\left(\frac{a_r \cdot A_m B_r}{R \cdot A_m A_r}\right). \quad \ldots\ldots\ldots (3)$$

The potential coefficients are, therefore, given by

$$p_{m.m} = 2 \log \{(R^2 - a_m^2)/R r_m\} \quad \ldots\ldots\ldots\ldots\ldots (4)$$

and

$$p_{m.r} = 2 \log \{(a_r \cdot A_m B_r)/R \cdot A_m A_r\}. \quad \ldots\ldots\ldots\ldots (5)$$

If R is large compared with a, so that a^2 can be neglected compared with R^2, these become

$$p_{m.m} = 2 \log (R/r_m) \quad \ldots\ldots\ldots\ldots\ldots (6)$$

and

$$p_{m.r} = 2 \log (R/A_m A_r). \quad \ldots\ldots\ldots\ldots\ldots (7)$$

Since r is small compared with a, $p_{m.m}$ will usually be greater than $p_{m.r}$ unless R is infinite, when in the limit $p_{m.m} = p_{m.r}$.

The capacity coefficients are given by

$$k_{m.r} = D_{m.r}/\Delta, \quad \ldots\ldots\ldots\ldots\ldots (8)$$

where

$$\Delta = \begin{vmatrix} p_{1.1} & p_{2.1} & \cdots, & p_{n.1} \\ p_{1.2} & p_{2.2} & \cdots, & p_{n.2} \\ \cdots & \cdots & \cdots & \cdots \\ p_{1.n} & p_{2.n} & \cdots, & p_{n.n} \end{vmatrix}$$

and $D_{m.r}$ is the minor of $p_{m.r}$ in this determinant.

Now these coefficients can be exactly represented by capacities $K_{0.m}$ between any wire m and earth, and $K_{m.r}$ between any two wires m and r, where

$$K_{0.m} = \sum_{r=1}^{r=n} k_{m.r} \quad \ldots\ldots\ldots\ldots\ldots (9)$$

and

$$K_{m.r} = -k_{m.r}. \quad \ldots\ldots\ldots\ldots\ldots (10)$$

Where there are a number of wires, the determinant of equation (8) becomes unmanageable except in special cases. It is still manageable, however, for a number of wires not greater than four.

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For a large number of wires, it is necessary to adjust them so that $p_{m.r}$ is small compared with $p_{m.m}$. In this case we have, neglecting small quantities of the second order,

$$k_{m.r} = -\frac{p_{m.r}}{p_{m.m} \cdot p_{r.r}} + \sum_{q=1}^{q=n} \left\{ \frac{p_{q.m} p_{q.r}}{p_{q.q} p_{m.m} p_{r.r}} \right\}, \quad \ldots\ldots\ldots \quad (11)$$

in which $q \neq m$ or r, and $p_{m.r}$ becomes negative if $r = m$.

Now suppose all the wires to be distributed along the surface of a cylinder of radius a concentric with the shield and parallel to its axis. Let $R/a = 1 + \rho$, where ρ is small. Equations (4) and (5) become

$$p_{m.m} = 2 \log \left\{ \frac{\rho (2 + \rho) a}{(1 + \rho) r_m} \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12$$

and

$$p_{m.r} = \log \left\{ 1 + \left[\frac{\rho (2 + \rho)}{2 (1 + \rho) \sin \frac{1}{2} \theta_{m.r}} \right]^2 \right\}, \quad \ldots\ldots\ldots \quad (13)$$

where

$$\theta_{m.r} = \theta_r - \theta_m.$$

Table I gives the values of $p_{m.r}$ for various values of ρ and $\theta_{m.r}$, from which we can see when $p^2_{m.r}$ can be neglected compared with $p^2_{m.m}$, since the value of $p_{m.m}$ will usually be between 5 and 15 :—

TABLE I.—Values of $p_{m.r}$.

$\theta_{m.r}$ =	5°.	10°.	15°.	20°.	30°.	45°.	60°.	90°.	120°.	180°.
$\rho = 0{\cdot}05$	0·814	0·267	0·134	0·078	0·035	0·016	0·010	0·006	0·004	0·002
0·10	1·76	0·791	0·429	0·248	0·128	0·062	0·035	0·019	0·012	0·010
0·15	2·44	1·27	0·764	0·497	0·260	0·126	0·076	0·039	0·029	0·021
0·20	2·93	1·67	1·09	0·746	0·409	0·208	0·128	0·064	0·045	0·035
0·30	3·63	2·33	1·64	1·21	0·718	0·396	0·254	0·136	0·091	0·070
0·40	4·13	2·81	2·07	1·59	1·01	0·592	0·388	0·215	0·147	0·114
0·50	4·53	3·17	2·41	1·91	1·28	0·784	0·531	0·298	0·211	0·163

If ρ is small we can expand equation (13) and obtain

$$p_{m.r} = \frac{\rho^2}{\sin^2 (\frac{1}{2} \theta_{m.r})} \left\{ 1 - \rho + \frac{\rho^2}{2} \left(\frac{5}{2} - \frac{1}{\sin^2 (\frac{1}{2} \theta_{m.r})} \right) + \ldots \right\}.$$

Hence, neglecting small terms of the second order,

$$K_{m.r} = \frac{1}{p_{m.m} p_{r.r}} \left\{ \frac{\rho^2 (1 - \rho)}{\sin^2 (\frac{1}{2} \theta_{m.r})} - \rho^4 \sum_{q=1}^{q=n} \left[\frac{1}{\sin^2 (\frac{1}{2} \theta_{m.q}) \sin^2 (\frac{1}{2} \theta_{r.q}) p_{q.q}} \right] + \ldots \right\}, \quad (14)$$

in which $q \neq m \neq r$.

If all the wires are of the same radius r, so that

$$p_{m.m} = p_{r.r} = \ldots \equiv p,$$

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we have

$$K_{0.m} = \frac{1}{p}\left\{1 + \frac{\rho^2(1-\rho)}{p}\sum_{q=1}^{q=n}\frac{1}{\sin^2(\frac{1}{2}\theta_{m.q})}\right\}, \quad \ldots\ldots\ldots \quad (15)$$

and

$$K_{m.r} = \frac{1}{p^2}\left\{\frac{\rho^2(1-\rho)}{\sin^2(\frac{1}{2}\theta_{m.r})} - \frac{\rho^4}{p}\sum_{q=1}^{q=n}\left[\frac{1}{\sin^2(\frac{1}{2}\theta_{m.q})\sin^2(\frac{1}{2}\theta_{r.q})}\right]\right\}, \quad \ldots \quad (16)$$

in which as before $q \neq m \neq r$.

If $p_{m.r}$ is sufficiently small, the last term in equation (16) can be neglected. It should also be noticed that for this case $K_{0.m}$ has the same value for all the wires.

III. Standard Resistance.

(a) *General Theorem for n Impedances in Series.*

Consider n impedances $Z_1, Z_2, \ldots Z_n$ (Fig. 2) connecting the points 1 and 2, 2 and 3, ... n and $(n+1)$ respectively. The point 0 will be assumed to be directly connected to the screen. Suppose the ends of these impedances to be interconnected by capacities, the

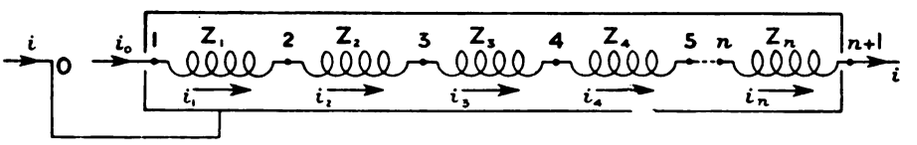

Fig. 2.

capacity connecting the points m and r being $C_{m.r}$. It is required to find the impedance of the whole, when the capacities are sufficiently small that their products can be neglected.

There will be two definitions of this impedance: one, Z_0, will be the ratio of the potential difference across the whole network to the current entering at the point 1, excluding the capacity-current to the screen, and Z_1 will be the same but including this capacity-current. In other words, the impedance will be Z_0 if the current is measured between the points 0 and 1, and Z_1 if the points 0 and 1 are on the same side of the point at which the current is measured with respect to any direct connexion with the source of E.M.F.

Let $i_1, i_2, \ldots i_n$ be the currents, and $V_1, V_2, \ldots V_n$ be potentials across $Z_1, Z_2, \ldots Z_n$ respectively. For brevity, we shall express the sum of these potential differences and impedances between the points m and r by $V_{m.r}$ and $Z_{m.r}$ respectively, so that

$$Z_{m.r+1} = \sum_{p=m}^{p=r}(Z_p) \quad \text{and} \quad V_{m.r+1} = \sum_{p=m}^{p=r}(V_p),$$

where $Z_{m.r}$ and $V_{m.r}$ are positive if $r > m$, and *vice versa*.

Now

$$V_m = Z_m \cdot i_m, \quad \ldots\ldots\ldots\ldots\ldots \quad (17)$$

$$i_0 = i_m + \sum_{p=1}^{p=m}\sum_{q=0}^{q=n+1}(V_{p.q}C_{p.q})\alpha, \quad \ldots\ldots\ldots \quad (18)$$

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and

$$i = i_m + \sum_{p=0}^{p=m} \sum_{q=0}^{q=n+1} (V_{p.q} C_{p.q})\,\alpha, \quad \ldots\ldots\ldots\ldots (19)$$

where i_0 and i are the currents entering the network (excluding and including, respectively, the earth-current), and $\alpha = \omega\sqrt{-1}$, the frequency being $\omega/2\pi$.

From these three equations, neglecting small quantities of the second order, we have

$$V_m/i_0 = Z_m - Z_m \sum_{p=1}^{p=m} \sum_{q=0}^{q=n+1} (Z_{p.q} C_{p.q})\,\alpha, \quad \ldots\ldots\ldots (20)$$

and

$$V_m/i = Z_m - Z_m \sum_{p=0}^{p=m} \sum_{q=0}^{q=n+1} (Z_{p.q} C_{p.q})\,\alpha, \quad \ldots\ldots\ldots (21)$$

in which $Z_{p.q}$ is negative if $p > q$, and *vice versa*.

Since

$$\sum_{m=1}^{m=n} \left\{ Z_m \sum_{p=1}^{p=m} \sum_{q=0}^{q=n+1} (Z_{p.q} C_{p.q})\,\alpha \right\} = \sum_{p=1}^{p=n+1} \sum_{q=p+1}^{q=n+1} (C_{p.q} Z^2_{p.q})\,\alpha - \sum_{m=1}^{m=n+1} (C_{0.m} Z_{0.m} Z_{m.n+1})\alpha,$$

the impedance of the network is given by

$$\begin{aligned} Z_0 &= \sum_{m=1}^{m=n} (V_m/i_0) \\ &= Z_{1.n+1} - \left[\sum_{p=1}^{p=n} \sum_{q=p+1}^{q=n+1} (C_{p.q} Z^2_{p.q}) - \sum_{p=1}^{p=n} C_{0.p} Z_{0.p} Z_{p.n+1} \right]\alpha, \quad \ldots\ldots (22) \end{aligned}$$

and

$$Z_1 = Z_{1.n+1} - \left[\sum_{p=1}^{p=n} \sum_{q=p+1}^{q=n+1} (C_{p.q} Z^2_{p.q}) + \sum_{p=2}^{p=n+1} (C_{0.p} Z_{0.p} Z_{1.p}) \right]\alpha, \quad \ldots (23)$$

where $Z_0 i_0$ is the potential of the point 1 below earth, and

$$i_0 = i\left[1 - \sum_{q=1}^{q=n+1} (Z_{0.q} C_{0.q})\,\alpha \right]. \quad \ldots\ldots\ldots (24)$$

Also, if i' be the current leaving the network,

$$i'/i = 1 - \sum_{q=0}^{q=n+1} (Z_{0.q} C_{0.q})\,\alpha. \quad \ldots\ldots\ldots (25)$$

(*b*) *Formulæ for Residual Inductance.*

Besides intercapacities, the wires have a small self-inductance L per unit length, given by Rosa* :

$$L = 2\,[\log (2l/r) - 1 + \mu/4], \quad \ldots\ldots\ldots (26)$$

where l is the length of the wire, r its radius and μ the permeability of the wire, that of the dielectric being unity.

* E. B. Rosa, "Self and Mutual Inductances of Linear Conductors," 'Bull. Bureau of Standards,' Vol. 4, p. 301, 1907.

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Also there are small mutual inductances M between the wires, given by*

$$M = \pm 2\,[\log(2l/d) - 1 + d/l], \quad \ldots\ldots\ldots \quad (27)$$

where d is the distance apart of the wires. The mutual inductance between the wires p and q, connected in the same sequence as in Fig. 1, will be negative if $(q - p)$ is odd, and positive if $(q - p)$ is even. In general the last term of equation (27) can be neglected.

It should be noticed that formulæ (4) and (5), (26) and (27) only hold for wires of circular cross-section.

Let us first assume that there are no capacities (which might be produced by terminals, etc.) at the ends of the wires. Consider a point P_m on the wire m (Fig. 1) distant x from m. Let Z_m, $M_{m.r}$, $K_{m.r}$, $K_{0.m}$ be the constants per unit length of this wire. Then

$$-\frac{di_{m+x}}{dx} = \sum_{r=0}^{r=n} (K_{m.r} V_{m+x.r+x})\,\alpha, \quad \ldots\ldots\ldots \quad (28)$$

where i_{m+x} is the current at P_m and $V_{m+x.r+x}$ is the potential difference between P_m and P_r on the same level as P_m on the rth wire. Also

$$-\frac{dV_{m+x}}{dx} = i_{m+x} Z_m + \sum_{r=1}^{r=n} (M_{m.r} i_{r+x})\,\alpha. \quad \ldots\ldots\ldots \quad (29)$$

Integrating equation (28) between the limits 0 and l, we have

$$i_m - i_{m+1} = \sum_{r=0}^{r=n} \left\{ K_{m.r} l i_1 \left[Z_{m.r} + (Z_r - Z_m)\frac{l}{2} \right] \alpha \right\},$$

where $Z_{m.r}$ is negative if $r < m$, and *vice versa*, i_m and i_{m+1} being the currents at the points m and $(m + 1)$ respectively.

By taking the sum of this between 1 and $(m - 1)$, we shall have an equation connecting i_m and i_1, by which i_m can be eliminated in the expression obtained on integrating equation (29). On rearranging, we finally have the impedances Y_0, Y_1 of the whole given by the general expressions :—

$$\begin{aligned} Y_0 &= \sum_{m=1}^{m=n} (V_m/i_0) \\ &= Z_{\cdot\ +1} + \Bigg[\sum_{m=1}^{m=n} \sum_{r=1}^{r=n} (M_{m.r} l) - \sum_{m=1}^{m=n} \sum_{r=m+1}^{r=n} (K_{m.r} l A_{m.r}) \\ &\qquad + \sum_{m=1}^{m=n} \{ K_{0.m} l\,[Z_{0.m} Z_{m+1.n+1} + Z_m l\,(\tfrac{1}{2} Z_{0.n+1} - \tfrac{1}{3} Z_m l)] \} \Bigg] \alpha, \end{aligned} \quad (30A)$$

and

$$\begin{aligned} Y_1 &= Z_{1.n+1} + \Bigg[\sum_{m=1}^{m=n} \sum_{r=1}^{r=n} (M_{m.r} l) - \sum_{m=1}^{m=n} \sum_{r=m+1}^{r=n} (K_{m.r} l A_{m.r}) \\ &\qquad - \sum_{m=1}^{m=n} \{ K_{0.m} l\,[Z_{0.m} Z_{1.m+1} - Z_m l\,(\tfrac{1}{2} Z_{0.1} - \tfrac{1}{3} Z_m l)] \} \Bigg] \alpha, \end{aligned} \quad (30B)$$

* E. B. Rosa, *loc. cit.*

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where

$$A_{m.r} = Z_{m+1.r+1}\left[Z_{m.r} + (Z_r - Z_m)\frac{l}{2}\right] + \frac{Z_{m.r}}{2}(Z_m - Z_r)\,l - (Z_m - Z_r)^2\frac{l^2}{6} \quad \text{if } (r-m) \text{ is even,}$$

and

$$A_{m.r} = Z_{m+1.r+1}\left[Z_{m.r} + (Z_r - Z_m)\frac{l}{2}\right] + \frac{Z_{m.r}}{2}(Z_m - Z_r)\,l - (Z_m - Z_r)^2\frac{l^2}{6} + \frac{1}{3}Z_m Z_r l^2 \quad \text{if } (r-m) \text{ is odd.}$$

Three special cases will be considered :—

(i) If all the wires are equal and have a resistance R each, we have

$$Y_0 = nR + \left[L_{1.n+1} + 2\sum_{m=1}^{m=n}\sum_{r=m+1}^{r=n} M_{m.r}\,l - \sum_{m=1}^{m=n}\sum_{r=m+1}^{r=n} R^2\,(K_{m.r}\,l A_{m.r}) + \sum_{m=1}^{m=n}\left\{K_{0.m}\,l\left[Z_{0.1}\,R\,(n-m+\tfrac{1}{2}) + R^2\left(\{n-m+1\}\,m - \frac{n}{2} - \frac{1}{3}\right)\right]\right\}\right]\alpha, \quad \ldots\ldots\ldots \quad (31)$$

where

$$A_{m.r} = (r-m)^2 + \tfrac{1}{3} \quad \text{if } (r-m) \text{ is odd,}$$

and

$$A_{m.r} = (r-m)^2 \quad \text{if } (r-m) \text{ is even.}$$

In the equation for Y_1 the last term is

$$-\sum_{m=1}^{m=n}\{K_{0.m}\,l\,[Z_{0.1}\,R\,(m-\tfrac{1}{2}) + R^2\,(m^2 - m + \tfrac{1}{3})]\}\alpha.$$

(ii) If the wires are all equally spaced, equation (31) can be written in the form :

$$Y_0 = nR + l\left[2\log\left\{\frac{(2l)^\gamma}{r^n}\right\} + \frac{m=n}{4}\sum_{m=2}\left\{(-1)^m\,(n-m+1)\left(\log d_{1.m} - \frac{d_{1.m}}{l}\right)\right\} + 2\left(\frac{n\mu}{4} - \gamma\right) - \sum_{m=1}^{m=n}\left\{(n-m+1)\,R^2 K_{1.m} A_{1.m} - K_{0.m}\left[Z_{0.1}\,R\,(n-m+\tfrac{1}{2}) + R^2\left(\{n-m+1\}\,m - \frac{n}{2} - \frac{1}{3}\right)\right]\right\}\right]\alpha, \quad (32)$$

where $\gamma = 0$ if n is even (which is the usual case) and $\gamma = 1$ if n is odd ; $A_{1.m}$ has the same value as in equation (31) ; $d_{1.r}$ is the distance between the 1st and the rth wire. The last term in the equation for Y_1 is the same as in case (i).

(iii) If $K_{0.m}$ has the same value (K_0, say) for all the wires, equation (32) can be further simplified. This will occur if all the wires are evenly spaced round the circumference of a cylinder concentric with the shield, or, again, if ρ in equation (15) is sufficiently small—*i.e.*, if the distance between the wires and the shield is small compared with its

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radius. In the latter case, however, there must be no bodies within the shield other than the wires used. Thus it would not necessarily hold if only a few of the wires were used and the rest connected to the shield or left floating.

With these conditions, equation (32) becomes

$$Y_0 = nR + l\left[2\log\left\{\frac{(2l)^\gamma}{r^n}\right\} + \frac{4}{m=2}\sum^{m=n}\left\{(-1)^m(n-m+1)\left(\log d_{1.m} - \frac{d_{1.m}}{l}\right)\right\}\right.$$
$$+ 2\left(\frac{n\mu}{4} - \gamma\right) - R^2\sum_{m=2}^{m=n}\{(n-m+1)K_{1.m}A_{1.m}\}$$
$$\left. + K_0\left(Z_{0.1}R\frac{n^2}{2} + R^2\frac{n^3}{6}\right)\right]\alpha. \quad . \quad . \quad . \quad (33)$$

The last term in the equation for Y_1 is

$$-K_0 l\left(Z_{0.1}R\frac{n^2}{2} + R^2\frac{n^3}{3}\right)\alpha.$$

To allow for the capacities $C_{m.r}$ between the terminals at the points m and r, and the earth-capacities $C_{0.m}$, $C_{0.r}$ between the terminals and earth, we can use equations (22) and (23), for the whole resistance can be considered to be made up of impedances of value (V_m/i) in series interconnected with capacities.

Hence, the total impedance is given by

$$Z_0 = Y_0 - \left[\sum_{p=1}^{p=n}\sum_{q=p+1}^{q=n+1}(C_{p.q}Z^2_{p.q}) - \sum_{p=1}^{p=n}(C_{0.p}Z_{0.p}Z_{p.n+1})\right]\alpha \quad . \quad . \quad . \quad (34)$$

and

$$Z_1 = Y_1 - \left[\sum_{p=1}^{p=n}\sum_{q=p+1}^{q=n+1}(C_{p.q}Z^2_{p.q}) + \sum_{p=2}^{p=n+1}(C_{0.p}Z_{0.p}Z_{1.p})\right]\alpha. \quad . \quad . \quad . \quad (35)$$

(c) *Application to Bridges.*

When a high impedance is put in a bridge a difficulty is met at once in defining the meaning of impedance, for, owing to capacity currents, the current entering any specific

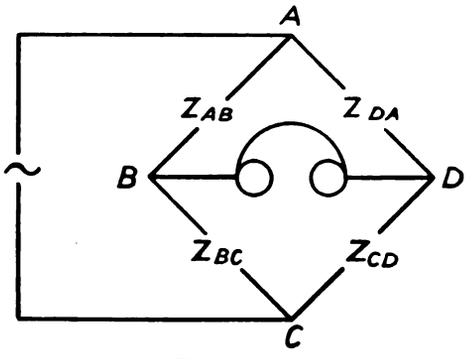

FIG. 3.

part of the bridge may not equal the current leaving it. We shall consider as a typical case the simple four-arm bridge (Fig. 3).

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The condition for balance is

$$V_{AB}/V_{BC} - V_{DA}/V_{CD}.$$

This can be written

$$(V_{AB} \cdot i_B)/(V_{BC} \cdot i_B) = (V_{DA} \cdot i_D)/(V_{CD} \cdot i_D),$$

or

$$Z_{AB}/Z_{BC} = Z_{DA}/Z_{CD}.$$

We see, therefore, that the impedance of any arm is defined as the ratio of the potential difference to the current at the end of the arm connected to the telephone.

High Resistance of small Self-Inductance for all Frequencies.—In deciding which formula is applicable for one of the impedance arms in any given case, the definition for Z_0 and Z_1 should be followed. It is only necessary, therefore, to see whether it is possible in the case of the impedance Z_{AB}, for instance, to follow the circuit from the point B to the point to which the screen of Z_{AB} is directly connected without passing through either of the points A or C. If A or C must be passed formula (22) should be used, otherwise formula (23) becomes applicable. The value of the impedance Z_1 of the equations must also be found. This impedance is defined as the ratio of the potential difference between the point B and the screen of the impedance Z_{AB} to the current at B. (It should be noted that this impedance may be negative.)

It will have been seen how important it is, in order that the bridge equation may hold, that the current at B in BA should be equal to that also at B in BC. To prevent earth capacity currents flowing in at B or D, which would destroy this equality, it is necessary that the points B and D should be at earth potential, but yet not directly connected to earth. It is in order to obtain this condition that a Wagner earth is used.

(*d*) *Case of a Single Loop.*

A special case of interest is that of two equal long parallel wires symmetrically situated with respect to the screen, with terminals at the end having capacities $C_{1.2}$ between them and $C_{0.1}$ to the screen. If R_0 is the total resistance of the two wires together, we have from equation (34)

$$Z_0 = R_0 + \left\{4l\left[\log(2a/r) + \tfrac{1}{4}\right] - R_0^2\frac{l}{3}K_{1.2} + R_0 l K_{0.1}\left(Z_{0.1} + \frac{R_0}{3}\right) + C_{0.1}R_0 Z_{0.1} - C_{1.2}R_0^2\right\}\alpha, \quad . \quad (36)$$

where r is the radius of the wires and $2a$ their distance apart. When the radius R of the screen is infinite, $K_{0.1} = C_{0.1} = 0$, and the equation becomes

$$Z_0 = R_0 + \left\{4l\left[\log(2a/r) + \tfrac{1}{4}\right] - \frac{R_0^2 l}{12\log(2a/r)} - R_0^2 C_{1.2}\right\}, \quad . \quad . \quad . \quad (37)$$

which is the formula given by S. L. BROWN.*

* S. L. BROWN, *loc. cit.*

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Let us consider how large the radius of the screen must be for this equation to hold.

If $Z_{0.1} = 0$—*i.e.*, one terminal is at earth potential—the last two terms in equation (35) become

$$-\frac{R_0^2 l}{6} \cdot \frac{\log\{(4\lambda^2+1)^2/[4\lambda\mu(4\lambda^2-1)]\}}{\log\{\mu(4\lambda^2-1)/(4\lambda^2+1)\}\log\{\mu(16\lambda^4-1)/16\lambda^2\}} - R_0^2 C_{1.2}, \quad . \quad (38)$$

where $\lambda = R/d$ and $\mu = d/r$.

Also, if $Z_{0\,1} = \frac{1}{2}R_0$, *i.e.*, the mid-point of the resistance is at earth potential, this becomes

$$-\frac{R_0^2 l}{12} \cdot \frac{1}{\log\{\mu(4\lambda^2-1)/(4\lambda^2+1)\}} - R_0^2(C_{1.2} + \tfrac{1}{2}C_{0.1}). \quad . \quad . \quad . \quad . \quad (39)$$

When λ is large, expression (39) is very nearly identical with the expression for an infinitely distant screen, for the measured capacity between the terminals is $(C_{1.2} + \frac{1}{2}C_{0.1})$. The same is not true for expression (38).

TABLE II.

Coefficient of $R_0^2 l$ in expression (36), $Z_{0.1} = 0$.

$\lambda =$	10	100	1000	10,000	∞
$\mu =$ 10	0	−0·0144	−0·020	−0·024	−0·036
= 100	0·0045	0	−0·0046	−0·0072	−0·018
= 1000	0·0096	0·0034	0	−0·0022	−0·012

In Table II the value of the coefficient of $R_0^2 l$ of the first term of expression (38) is given for various values of λ and μ. It will be seen that, as μ will usually lie between 100 and 1000, this coefficient will require an enormous value of λ to approach its limiting value. Thus, if $\mu = 100$ and $d = 1$ cm., this coefficient only reaches half its limitin value when all objects connected to earth are removed to a distance of 1 kilometre.

A similar result is obtained when Z_1 of equation (35) is taken instead of Z_0. Hence S. L. BROWN'S formula only applies when the mid-point of the resistance is at earth potential; but we saw in Section III (*c*) that in bridges $Z_{0.1}$ was either zero or positive, so that the conditions necessary for BROWN'S formula do not obtain. Experimental agreement was obtained because only differences were being measured, and the effect of the closeness of the earth would be substantially the same in both cases.

IV. STANDARD RATIO ARMS.

(*a*) *General Theorem for Two Impedances in Parallel.*

Consider n impedances in parallel with n' impedances (n does not necessarily equal n') interconnected by capacities at all points between the impedances. Let $Z_{0.A}$ be the impedance between the points a and a' and 0, which is connected to earth. This

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impedance can be split up into two impedances in parallel, $Z_{0.A}(i_0 + i_0')/i_0$ and $Z_{0.A}(i_0 + i_0')/i_0'$, carrying the currents i_0 and i_0' respectively (Fig. 4).

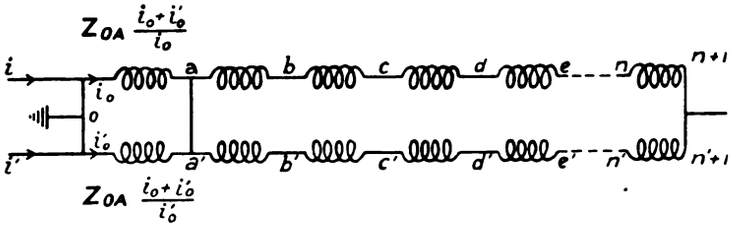

FIG. 4.

Using the same notation as previously, we have

$$V_m = Z_m i_m,$$

and

$$i_0 = i_m + \sum_{p=a}^{p=m}\sum_{q=0}^{q=n+1}(C_{p.q}V_{r.q})\alpha + \sum_{p=a}^{p=m}\sum_{q=a'}^{q=n'+1}(C_{p.q}V_{p.q})\alpha \quad . \quad . \quad . \quad . \quad (40)$$

and

$$i = i_m + \sum_{p=0}^{p=m}\sum_{q=0}^{q=n+1}(C_{p.q}V_{p.q})\alpha + \sum_{p=a}^{p=m}\sum_{q=a'}^{q=n'+1}(C_{p.q}V_{p.q})\alpha, \quad . \quad . \quad . \quad (41)$$

and similarly for i_0' and i'.

Now,

$$\sum_{m=0}^{m=n}(V_m) = \sum_{m=0}^{m=n'}(V_m).$$

Writing P_0 and P_0' for the expressions for the impedances as given in Section III, for each of the two networks in parallel (these expressions will, of course, not contain the terms involving the capacities or mutual inductances between one network and the other), we have

$$\frac{i_0}{i_0'} \equiv \frac{Z_0'}{Z_0} = \sigma\left\{1 + \left[\frac{P_0'}{Z_{a'.n'+1}} - \frac{P_0}{Z_{a.n+1}} + \sum_{p=a}^{p=n+1}\sum_{q=a'}^{q=n'+1} C_{p.q}(Z_{p.n+1} + Z_{q.n'+1})\left(\frac{Z_{a'.q}}{Z_{a'.n'+1}} - \frac{Z_{a.p}}{Z_{a.n+1}}\right)\right]\alpha\right\}, \quad . \quad . \quad . \quad (42)$$

where $\sigma = Z_{a'.n'+1}/Z_{a.n+1}$ and Z_0, Z_0' are the effective impedances of each of the two networks in parallel. P_0 and P_0' are given by equation (22), where $Z_{0.a} = Z_{0.A}(1 + 1/\sigma)$ and $Z_{0.a'} = Z_{0.A}(1 + \sigma)$.

Also

$$\frac{i}{i'} \equiv \frac{Z_1'}{Z_1} = \sigma\left\{1 + \left[\frac{P_1'}{Z_{a'.n'+1}} - \frac{P_1}{Z_{a.n+1}} + \sum_{p=a}^{p=n+1}\sum_{q=a'}^{q=n'+1} C_{p.q}(Z_{p.n+1} + Z_{q.n'+1})\left(\frac{Z_{a'.q}}{Z_{a'.n'+1}} - \frac{Z_{a.p}}{Z_{a.n+1}}\right)\right]\alpha\right\}, \quad . \quad . \quad . \quad (43)$$

where P_1 and P_1' are given by equation (23)

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P_0, P_0', Z_0, Z_0' of equation (42) are the same as P_1, P_1', Z_1, Z_1' of equation (43), except that the former refer the impedances to the current entering the network including the earth capacity current, while the latter refer to the current entering the network excluding the earth capacity current.

(b) Formulæ for Ratio.

We shall now apply the last result to the calculation of the true ratio of the standard used as ratio arms.

As before, we shall assume at first that there are no capacities at the ends of the wires. Consider a point P_m on the wire m (Fig. 1) distant x from m. With the same notation as before, we have

$$-\frac{di_{m+x}}{dx} = \sum_{r=0}^{r=n} (K_{m.r} V_{m+x.r+x})\,\alpha + \sum_{r=a}^{r=n'} (K_{m.r} V_{m+x.r+x'})\,\alpha \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and

$$-\frac{dV_{m+x}}{dx} = i_{m+x} Z_m + \sum_{r=a}^{r=n} (M_{m.r} i_{r+x})\,\alpha + \sum_{r=a'}^{r=n'} (M_{m.r} i_{r+x})\,\alpha, \quad . \quad . \quad . \quad . \quad (45)$$

where $M_{m.r}$ is given by equation (27). It should be noticed that, if the current enters the two arms at the same end, the mutual inductance between one wire m in one arm and one wire r in the other is negative if $(r - m)$ is odd and positive if $(r - m)$ is even.

To make the conditions more general, we shall assume the arms to be connected in series with external impedances $Z_{A.a}$ and $Z_{A.a'}$, the standard containing n and n' wires respectively, numbered from a to n and a' to n'. These external impedances will be assumed to have zero mutual inductance with the standard, and there will, of course, be no capacity between them and the standard, since the latter is completely screened.

From equations (44) and (45), by integrating between the limits 0 and l and neglecting small terms of the second order, we obtain a general expression for $(Z_{A.n'+1}/Z_{A.n+1})$. The complete expression is lengthy, and, as the only case of importance is that when all the wires are equal and of resistance R, the simplified results in these conditions are given by

$$\frac{Z_{A.n'+1}}{Z_{A.n+1}} = \frac{Z_{A.a'} + Q'}{Z_{A.a} + Q}\left[1 + R^2 \sum_{p=a}^{p=n}\sum_{q=a'}^{q=n'} \left\{K_{p.q} l \left(\frac{B'_{p.q}}{Z_{A.a'} + Q'} - \frac{B_{p.q}}{Z_{A.a} + Q}\right)\alpha\right\} + \left(\frac{1}{Z_{A.a'} + Q'} - \frac{1}{Z_{A.a} + Q}\right) \sum_{p=a}^{p=n}\sum_{q=a'}^{q=n'} (M_{p.q}\alpha)\right], \quad . \quad . \quad . \quad . \quad (46)$$

where Q and Q′ are the expressions of the impedances as given in equations (31), (32), and (33) for each of the two ratio arms of the standard, and

$$B'_{p.q} = (n' - q)(q - \tfrac{1}{2}) + q - \tfrac{1}{2} \quad \text{if } q - p \text{ is odd,}$$
$$= (n' - q)(q - \tfrac{1}{2}) + q - \tfrac{2}{3} \quad \text{if } q - p \text{ is even,}$$

and

$$B_{p.q} = (n - p)(p - \tfrac{1}{2}) + p - \tfrac{1}{2} \quad \text{if } q - p \text{ is odd,}$$
$$= (n - p)(p - \tfrac{1}{2}) + p - \tfrac{2}{3} \quad \text{if } q - p \text{ is even.}$$

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To allow for the capacities between the terminals at the ends of the wires, we can now use equations (42) and (43), in which we substitute $(Z_{A.n'+1}/Z_{A.n+1})$ as given in equation (46) for σ.

(c) *Application to Bridges.*

The same arguments can be applied regarding the application to ratio arms in bridges as was used when a non-inductive resistance was considered.

V. Standard Potential Divider.

(a) *General Formulæ.*

In this case we require to find the ratio of the voltage between any two points r and s and the total voltage across the resistance. It will be supposed that no current is taken at the points r and s, but the capacity of the leads, etc., connected to those two points must be taken into account in the usual way.

From equation (20) we have

$$\frac{V_{r.s}}{i_0} = \sum_{m=r}^{m=s-1} (V_m/i_0)$$

$$= Z_{r.s} - \left[Z_{r.s} \sum_{p=1}^{p=r} \sum_{q=0}^{q=n+1} (C_{p.q} Z_{p.q}) + \sum_{p=r+1}^{p=s-1} \sum_{q=0}^{q=n+1} (C_{p.q} Z_{p.q} Z_{p.s}) \right] \alpha, \quad . \quad . \quad (47)$$

giving the potential difference between the points r and s when the potential divider is assumed to be made up of n impedances in series.

By putting $r = 1$ and $s = n + 1$, we obtain the potential difference across the whole. From this, we have

$$\frac{V_{r.s}}{V_{1.n+1}} = \frac{Z_{r.s}}{Z_{1.n+1}} \left\{ 1 - \left[\sum_{p=2}^{p=r} \sum_{q=0}^{q=n+1} (C_{p.q} Z_{p.q}) + \frac{1}{Z_{r.s}} \sum_{p=r+1}^{p=s-1} \sum_{q=0}^{q=n+1} (C_{p.q} Z_{p.q} Z_{p.s}) \right.\right.$$

$$\left.\left. - \frac{1}{Z_{1.n+1}} \sum_{p=2}^{p=n} \sum_{q=0}^{q=n+1} (C_{p.q} Z_{p.q} Z_{p.n+1}) \right] \alpha \right\}.$$

. . . . . (48)

Now suppose the tapping points are at points distant x and y from r and s respectively. Let us call these points $(r + x)$ and $(s + y)$.

From equations (28) and (29) it can be shown that

$$\frac{V_{r+x.s+y}}{i_1} = Z_{r+x.s+y} + \sum_{p=1}^{p=n} \left\{ M_{s.p} y - M_{r.p} x + \sum_{q=r}^{q=s-1} (M_{p.q} l) \right\} - \sum_{p=r-1}^{p=s-1} \sum_{q=p+1}^{q=n} (K_{p.q} l A_{p.q})$$

$$- \sum_{p=1}^{p=r-1} K_{0.p} l \left\{ Z_{0.p} Z_{p+1.r} + Z_p l \left(\tfrac{1}{2} Z_{0.r} - Z_p \frac{l}{3} \right) \right\}$$

$$- Z_r x \sum_{p=0}^{p=n} (\tfrac{1}{2} K_{r.p} x B_{r.p})$$

$$+ \sum_{p=1}^{p=s+1} K_{0.p} l \left\{ Z_{0.p} Z_{p+1.s} + Z_p l \left(\tfrac{1}{2} Z_{0.s} - Z_p \frac{l}{3} \right) \right\}$$

$$+ Z_s y \sum_{p=0}^{p=n} (\tfrac{1}{2} K_{s.p} y B_{s.p}). \quad . \quad . \quad (49)$$

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where

$$A_{p.q} = \left[Z_{p.q} + (Z_q - Z_p)\frac{l}{2}\right] Z_{p+1.q+1} + \tfrac{1}{2}Z_{p.q}(Z_p - Z_q)\,l - \frac{l^2}{6}(Z_p - Z_q)^2 \quad \text{if } (q-p) \text{ is even,}$$

and

$$A_{p.q} = \left[Z_{p.q} + (Z_q - Z_p)\frac{l}{2}\right] Z_{p+1.q+1} + \tfrac{1}{2}Z_{p.q}(Z_p - Z_q)\,l - \frac{l^2}{6}(Z_p + Z_q)^2 + Z_pZ_ql^2 \quad \text{if } (q-p) \text{ is odd.}$$

Also

$$B_{r.p} = Z_{r.p} + (Z_p - Z_r)\frac{x}{3} \qquad \text{if } (p-r) \text{ is even,}$$

and

$$B_{r.p} = Z_{r.p+1} - (Z_p + Z_r)\frac{x}{3} \qquad \text{if } (p-r) \text{ is odd.}$$

(For $B_{s.p}$ put y instead of x and s instead of r.)

The general expression for $(V_{1.n+1}/i_1)$ is given by equation (30B). By dividing equation (49) by this, we obtain the ratio $(V_{r+x.s+y}/V_{1.n+1})$.

To allow for the capacities of the terminals, this ratio must be substituted for

$$(Z_{r.s}/Z_{1.n+1})$$

in equation (48).

This may appear very laborious, but at the worst when there are many wires it involves only very simple arithmetic. The expression $(V_{r+x.s+y}/i_1)$ is usually simplified by the fact that all the wires will in general be of the same resistance R, and very often the earth capacity terms will be large compared with the other capacity terms. Besides this, in the usual case, $r = 1$ and $x = y = 0$.

When all the wires are of resistance R, the values of $A_{p.q}$ and $B_{p.q}$ simplify to

$$A_{p.q} = [(q-p)^2 + \tfrac{1}{3}]\,R^2 \qquad \text{if } (p-q) \text{ is odd,}$$

and

$$A_{p.q} = (q-p)^2 R^2 \qquad \text{if } (p-q) \text{ is even.}$$

Also

$$B_{r.p} = (p - r + x/3)R \qquad \text{if } (p-r) \text{ is odd,}$$

and

$$B_{r.q} = (p-r)\,R \qquad \text{if } (p-r) \text{ is even.}$$

(*b*) *Application.*

Apart from the general use of the potential divider at audio and radio frequencies, the standard may be of special use for obtaining small known voltages at radio frequencies, the voltage across the whole resistance being measured by an electrostatic voltmeter. When very low voltages are required, it is easier to connect a low variable resistance in

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series with the standard and take off the desired voltage across that resistance. It is, of course, assumed that the frequency is never so high that terms containing squares of the capacities cannot be neglected.

We have seen that, to obtain a calculable solution, it was necessary to have the shield fairly close to the wires. When dealing with very high voltages, this may bring about difficulty due to corona effect and sparking directly to the screen. If a number of standards are used in series this might largely be obviated by connecting the screen to one of its terminals, so that the potential difference in any one of them is only a fraction of the total. It will then become necessary to measure or calculate the capacity of the various screens to earth. For calculation, the simplest case would be to place all the standards in a line and surround the whole with a concentric metal shield (which can be a simple wire mesh), when the capacity of each standard to earth will be

$$\frac{l}{2 \log (r_1/r_2)},$$

where r_1 and r_2 are the radii of the external screen and of the standard respectively, l being the length.

VI. General Construction.

(*a*) *Design.*

The case of a 10,000 ohm standard will be considered: if No. 47 manganin wire, radius 0.0025 cm., be used, it may consist of 40 wires about 106 cms. long. The angle subtended at the centre by two consecutive wires will thus be 9°. From Table I, ρ must therefore be of the order of 0.1 or less. Suitable values are :—

Internal radius of screen ...	...	=	2 inches.
Distance of wires from centre	...	=	1.8 ,,

Hence

$$\rho = 0.111.$$

If there are no lumped capacities at the terminals, the total impedances of the 40 wires in series will be, from equation (33),

$$Y_0 = 10{,}000 + 0.00935\alpha$$

and

$$Y_1 = 10{,}000 - 0.0201\alpha.$$

To allow this standard to be used for a number of values of resistance, the wires must be connected to terminals at one end; at the other end they would simply be held taut by springs, to which they would not be electrically connected. In order to keep the wires in position along their whole length, they could be supported at intervals by ebonite bridges.

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A suitable arrangement of the terminals is shown in Fig. 5. The large dots represent the terminals (which may take the form of a split wire) to which the wires are fixed. The resistance wires 1 and 2, 3 and 4, etc., are connected at the further end, and the wires 2 and 3, 4 and 5, etc., can be connected by thin copper wires at the terminals. The other circle of terminals is connected up by a thin copper wire and serves as a connecting link between the last wire and terminal A.

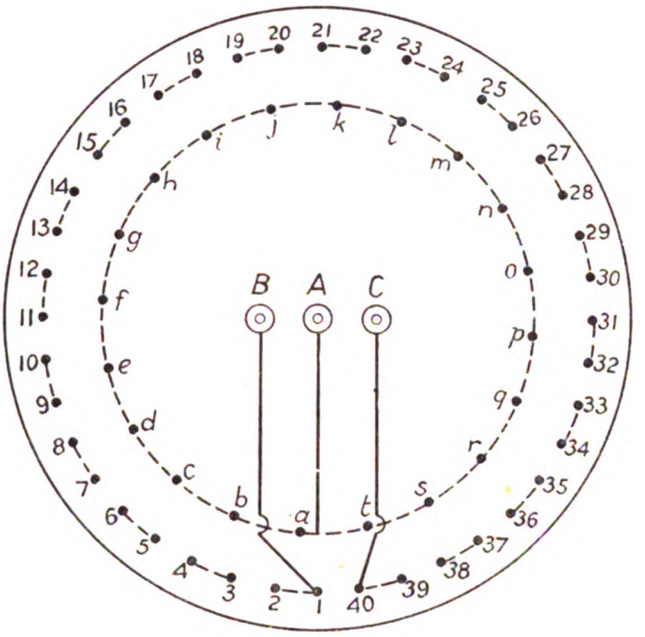

FIG. 5.

Thus, if terminal 10 is connected to *e*, the resistance will be 2,500 ohms. All the other wires will be connected together and to earth. In such a case the values of the capacities to earth will be slightly altered, because of the other earthed wires and terminals. The capacity to earth of any wire will be its usual value plus the sum of the capacities between the wire under consideration and all the earthed wires.

The correction due to the return path *a*, *b*, *c*, *d*, *e*, ... can be allowed for, if the capacities are known and the mutual inductance is negligible.

Three terminals ABC are shown, to allow for the use of the standard as ratio arms, one arm starting at wire 1 and the other at wire 40. In this case it would be advisable to cut the return wire *a*, *b*, *c* ... at some point between the two arms.

The measurement of the terminal capacities will be somewhat lengthy, owing to their large number. Fortunately, the work is considerably shortened owing to the symmetry of the arrangement. It is necessary to know the capacity of any terminal to the screen, to any other terminal, and to the return wire.

Connect all the wires except the *m*th and *r*th together. Call this A and the screen S. By connecting *m*, *r*, S, and A in various ways we can find their capacity thus:

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Connecting A and r to S and finding the capacity to m, we have

$$C_{Am} + C_{mr} + C_{ms} = C_1.$$

Connecting r to S and A to m and finding the capacity between them, we have

$$C_{Ar} + C_{AS} + C_{mS} + C_{mr} = C_2.$$

Connecting m and r to S and finding the capacity to A, we have

$$C_{AS} + C_{Am} + C_{Ar} = C_3.$$

Connecting A to S and m to r and finding the capacity between them, we have

$$C_{mS} + C_{rS} + C_{Am} + C_{Ar} = C_4.$$

Therefore, since m and r are symmetrical,

$$\begin{aligned} C_{Ar} = C_{Am} &= \tfrac{1}{2}(C_1 + C_3 - C_2), \\ C_{AS} &= C_2 - C_1, \\ C_{rS} = C_{mS} &= \tfrac{1}{2}(C_4 + C_2 - C_1 - C_3), \\ C_{mr} &= C_1 - \tfrac{1}{2}C_4. \end{aligned}$$

The value of C_1 must be found for all values of r. From this, since C_4 is from symmetry independent of r, we can find the values of C_{mr} for all values of r, and, since

$$C_{mS} = C_{0m} + \sum_{p=1}^{p=n} (C_{mp}) - C_{mr},$$

where C_{0m} is the capacity to the screen alone, we can find C_{0m}.

The intercapacities and capacities to earth of the terminals A, B and C will have to be found in a similar way.

It will be seen that the capacities of the wires can be measured, instead of calculated, when the wires are not permanently connected together. When this is done, the limitation of having to screen close to the wire vanishes. This limitation also disappears if the number of wires is not greater than four.

(*b*) *Accuracy.*

The intercapacities of the terminal can be measured by the method of HARTSHORN,* by which these small capacities can be measured to within 0.02 $\mu\mu$ F. The error in the residual inductance due to this will, in a 10,000-ohm standard, be of the order of 0.005 α.

* HARTSHORN, 'N.P.L. Collected Researches,' Vol. 21, Paper 7, p. 113 (or 'Proc. Phys. Soc. Lond.,' Vol. 36, p. 399, 1924).

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The main term of the residual inductance is due to the earth capacities K_{0m}, the other terms being but a few per cent. of this one. If errors of 10 per cent. occur in measuring the radius of the wires and the distance of the wires from the shield, the error in K_{0m} will be, for the case under consideration, of the order of 10 per cent. for the worst case, when all the errors are cumulative. Hence the maximum possible error in calculating the residual inductance of the standard, taking into account the error due to the assumption that the wire is infinitely long, is about 11 per cent.

The accuracy of calculation depends largely on the accuracy of the measurement of the radius of the wire and of the distance between the wire and the shield, each representing a possible error of about 5 per cent. in the residual inductance for the case considered. To reduce the quadrature component, it would be necessary to use a larger shield and to obtain the values of K_{0m}, K_{mr} by measurement.

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VII.

A METHOD OF MEASURING VERY SMALL CAPACITIES

ALSO A

NOTE ON THE CAPACITIES OF SMALL AIR CONDENSERS

BY

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ABSTRACTS.

A Method of Measuring Very Small Capacities.

A bridge arrangement is described suitable for the measurement of very small capacities (of the order of 1 micro-microfarad and less) at telephonic frequencies. Means of securing (1) great sensitivity, (2) fineness of adjustment, and (3) elimination of capacities to earth and capacities due to the presence of connecting leads, are indicated, so that it is possible to measure the true intercapacity between any two conductors, *e.g.*, the electrodes of a thermionic valve. As a test of the method, it is used to measure the capacity between two steel balls of 1 cm. radius at distances apart of 2·25 to 8 cms.

Note on the Capacities of Small Air Condensers.

The most general case of a condenser is considered, *i.e.*, any two conductors in a dielectric medium, and taking Maxwell's equations for the potentials and charges, it is pointed out that several definitions of capacity are possible, giving slightly different values in any one case. One of these is independent of earth effects.

In order to consider the case of a condenser carrying alternating current, the general equations of currents and potentials are written down. In general the effective capacity is seen to depend on the potential distribution and earth effects. The desirability of making the effective capacity depend on the geometry of the system and the nature of the dielectric only is pointed out, and the conditions necessary for this are deduced. Examples of the practical realisation of the conditions are given. The importance of these considerations in the case of shielded air condensers is indicated.

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A METHOD OF MEASURING VERY SMALL CAPACITIES.

By L. HARTSHORN, B.Sc., A.R.C.S., D.I.C., Senior Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 36, 1924.]

For the calibration of condensers of all the more usual ranges required for general laboratory work the Carey Foster bridge is admirable. At the National Physical Laboratory it is used for the determination of the capacities of condensers ranging from 10 microfarads to something like 50 micro-microfarads, and these small values may be determined with care within about $\pm$ 0.1 micro-microfarad. In radio-frequency circuits it often happens that there are certain very small capacities, a knowledge of which is of considerable importance, *e.g.*, the capacity between the electrodes of a thermionic valve. These capacities may only be of the order of 1 micro-microfarad, or even less, so that they cannot be determined with any certainty on the Carey Foster bridge. The present paper describes a method which has been used with some success for measuring such small capacities.

In the general testing of variable air condensers, it was noticed that bridges of the Wien type were much more sensitive than the Carey Foster bridge, and for this reason it is often convenient to use such bridges when it is merely desired to compare two variable air condensers. Provided a substitution method is used, and care is taken about the arrangement of the screens of the condensers and the earthed point of the bridge, the Schering modification of the Wien bridge is capable of great precision. Thus in setting out to measure very small capacities, it is natural to turn to a bridge of this type. The difficulty which then arises is the need of a variable standard of capacity which is capable of extremely fine adjustment, and which has a very open scale already calibrated. For such a standard the writer has used two condensers in series, the capacity of one, C, being large compared with that of the other, c (Fig. 1). If K is the capacity of the combination, then $K = cC/(C + c)$. Thus

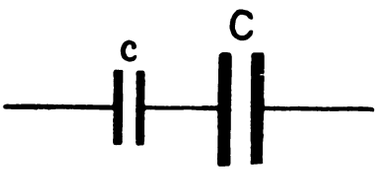

Fig. 1.

$$dK = \left(\frac{c}{C + c}\right)^2 dC.$$

If now $c/(C + c)$ is arranged to be 0.1, then $\delta K = 0.01\delta C$. Thus if C is changed by 1 $\mu\mu$F, then the capacity of the combination is changed by one-hundredth of this

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amount. If C is an ordinary variable air condenser calibrated to read micro-micro-farads directly, then obviously by putting a suitable small condenser in series with it, the scale of C may be made direct-reading in hundredths or even thousandths of micro-microfarads for small changes of capacity of the combination. The change in C must be so small that the value of $c/(C + c)$ is not sensibly altered if the arrangement is to be direct-reading. If the change in this ratio cannot be neglected, the change in capacity of the combination must be calculated from

$$\Delta K = \frac{c^2 (C_1 - C_2)}{(C_1 + c)(C_2 + c)},$$

where C_1 and C_2 are the two readings of C, *i.e.*, in calculating the magnification factor $C^2/(C + c)^2$ the geometric mean of the two values of $(C + c)$ must be used.

The Bridge Used.

A bridge found very convenient for this purpose is that due to SCHERING. The arrangement is shown diagrammatically in Fig. 2. The condenser to be tested is C_5. By means of a pair of rigid leads l, it may be inserted in arm No. 1 of the bridge in parallel

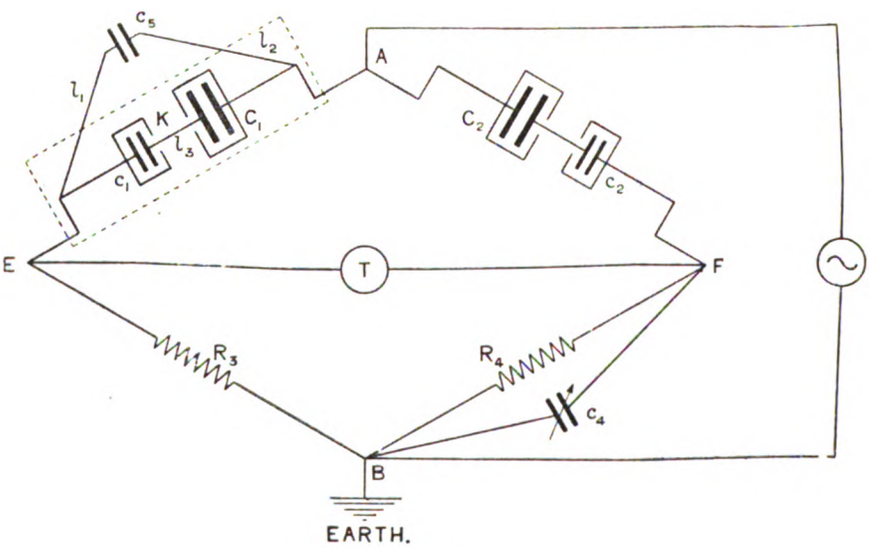

FIG. 2.

with the combination standard K made up of the two condensers c_1 and C_1. Arm No. 2 of the bridge is practically a duplicate of arm No. 1, the idea being to make the bridge as symmetrical as possible. Arms 3 and 4 consist of non-inductive resistances, each of 5,000 ohms, and in parallel with one of these is a small variable condenser C_4, which is for the phase angle adjustment. R_3, R_4 and C_4 are all enclosed in metal screens which are connected to the earthed point B of the bridge. The screens of the large condensers

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C_1 and C_2 are connected to the point A. Their capacities to earth are thus merely a shunt across the whole bridge, and do not affect the balance. The screens of c_1 and c_2, which are much smaller, are connected to the points E and F respectively. Their capacities to earth are shunted across the arms R_3 and R_4. They thus are balanced one against the other. These condensers must be placed as far away from the observer as possible, so that their capacities to earth are as little as possible affected by his movements. These earth capacities can, of course, be entirely eliminated (being located at the corners of the bridge) by WAGNER'S earthing device, but this is an additional complication, which is not desirable. In the arrangement shown, since the impedances of the arms 1 and 2 are enormous compared with those of 3 and 4, the points E and F are not much above earth potential, and thus the earth capacities at these points are only of comparatively small effect.

The extreme fineness of adjustment referred to above would, of course, be of no advantage unless the bridge could be made correspondingly sensitive. It is comparatively easy to do this with the Schering bridge. The voltage applied to the points A and B is simply raised until the sensitivity is sufficient. The current carried by the resistances R_3 and R_4 has to pass across the very high impedances in arms 1 and 2 of the bridge, so that this current is always small. The voltage which can be applied is limited only by what the condensers will stand. Several hundreds of volts may be used quite safely in nearly all cases. The bridge takes very little power, and thus a very small transformer is sufficient to step up the voltage to the required amount. In practice the leads to the bridge were connected to a coil of several henries inductance, and this was brought near to the inductance coils of an oscillating valve set until the required voltage was obtained.

The capacities c_1 and C_1 include the capacity between the lead l_3 and the screen of the condensers. This must not be lost sight of in making the calculations. The lead should be of fine wire, and this lead capacity should be included when c_1 and C_1 are calibrated. In any case this lead capacity will be very small, and will not depend very much on the length of lead beyond 10 cms., say. The capacity between the two screens is merely added to the total capacity in arm 1 and does not affect the ratio $(C + c)/c$.

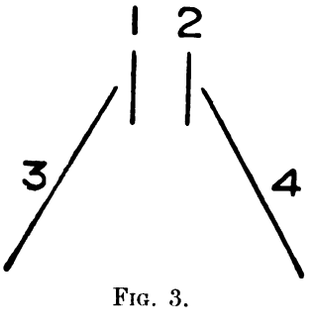

FIG. 3.

Effect of the Leads.

In Fig. 3, let 1 and 2 be the plates of the small condenser, the capacity of which is to be determined, and let 3 and 4 represent the leads used to connect the condenser to the bridge. This system of four conductors possesses ten capacity constants—viz., C_{1e}, C_{2e}, C_{3e}, C_{4e}, the earth capacities, and C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}, the intercapacities. The quantity to be determined is C_{12}. It is to be noted that the bridge arrangement described above measures intercapacities only. The earth

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capacities on the A side of arm 1 may be regarded as a shunt across the whole bridge, having no effect on the balance, while those on the E side act as a shunt across R_3, and are balanced by adjusting C_4. Thus, in considering the capacity balance we may ignore all capacities to earth. The procedure to be adopted is as follows :—

(*a*) Connect 1 and 2 to 3 and 4 respectively, and note the bridge reading when balance is obtained. The effective capacity between the leads is now the intercapacity of the two combined conductors, 1, 3 and 2, 4—*i.e.*, of—

$$C_{12} + C_{14} + C_{32} + C_{34}.$$

(*b*) Break the connexion 1–3 without disturbing the lead appreciably, connect 1 to earth, and again take the balance reading. The capacities C_{12}, C_{14} have now become earth capacities, and are thus eliminated, and the effective capacity between the lead is

$$C_{32} + C_{34}.$$

(*c*) Again make the connexion 1–3, break the connexion 2–4 and earth 2. The balance reading now corresponds to a capacity between the leads

$$C_{14} + C_{34}.$$

(*d*) Break both connexions 2–3 and 3–4, and earth both 1 and 2. All the capacities except C_{34} now become earth capacities, and thus the effective capacity between the leads is now C_{34}.

Taking the differences of the readings (*a*) — (*d*), (*b*) — (*d*), (*c*) — (*d*) we obtain the values of $C_{12} + C_{14} + C_{23}$, C_{23}, and C_{14} respectively, and thus C_{12} may be determined. The positions of the conductors must not be disturbed in taking any of the readings, or some of the capacities may be changed. For example, the capacity between the leads C_{34} is influenced by the presence of the plates 1 and 2, and thus the leads reading must be taken with 1 and 2 in position.

Example.

As a test of the method it was used to measure the capacity between two steel balls of 1 cm. radius at various distances apart. Each of the balls was supported on a small pillar of paraffin wax. The condensers c_1, C_1, c_2, C_2, consisted of small fixed mica condensers. Such condensers have the advantage of very small bulk, and thus small capacities between their shields and the leads, and between shields and earth. The values were :—

$c_1 = 0.000974$ microfarad.
$C_1 = 0.010116$,,

In parallel with C_1 was a variable air condenser which allowed of the necessary adjust-

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ments being made. The mean reading of this condenser was approximately 1,100 micro-microfarads, thus the total value of C_1 was about 0.0112 microfarad, and the magnification factor $(C_1 + c_1)^2/c_1^2$ about 160. The leads to the balls were copper wire about No. 30 gauge. Contact was made simply by allowing the ends to touch the polished surface of the balls. The values obtained are given in the table.

TABLE.—Capacity between Steel Balls of 1 cm. Radius.

Distance between centres. cms.	Capacity in micro-microfarads.	
	Observed.	Calculated.
2·25	0·71	0·69
3·0	0·45	0·43
4·0	0·32	0·30
6·0	0·19	0·19
8·0	0·15	0·14

The capacities C_{14} and C_{23} between the balls and the leads varied from 0.18 to 0.12 micro-microfarad, depending on the distance apart. The calculated values were obtained by using RUSSELL'S* formula for the mutual capacity coefficients of two spherical conductors. The agreement between the observed and calculated values is as close as can be expected. The observed values are on the whole a trifle higher than the calculated ones. This is no doubt due to the presence of the paraffin wax pillars used to support the balls.

(The value given by RUSSELL for the "capacity between two spheres" of 1 cm. radius and 4 cms. apart is 0.74 $\mu\mu$F, which is much larger than the above value. This is because the intercapacity between the spheres which is measured by the bridge is not the same as the "capacity between spheres," defined in RUSSELL'S paper. This quantity is a function of the earth capacities of the spheres as well as of the intercapacity. The point is fully dealt with in the "Note on the Capacities of Small Air Condensers," which follows.)

NOTE ON THE CAPACITIES OF SMALL AIR CONDENSERS.

BY L. HARTSHORN, B.Sc., A.R.C.S., D.I.C., SENIOR ASSISTANT IN THE ELECTRICITY DEPARTMENT.

[Reprinted from the 'JOURNAL OF SCIENTIFIC INSTRUMENTS,' Vol. 1, 1924.]

WITH the increasing importance of radio-frequency measurements, variable air condensers are becoming more and more widely used as laboratory standards. Such condensers are usually almost completely enclosed in a metallic screen, and are often

* A. RUSSELL, 'Proc. Phys. Soc. Lond.,' Vol. 35, p. 10, 1922.

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engraved "calibrated with screen earthed" or "calibrated with moving vanes connected to screen." The calibration may vary considerably with the conditions of test and it becomes necessary to enquire into the meaning of the results obtained under any given conditions.

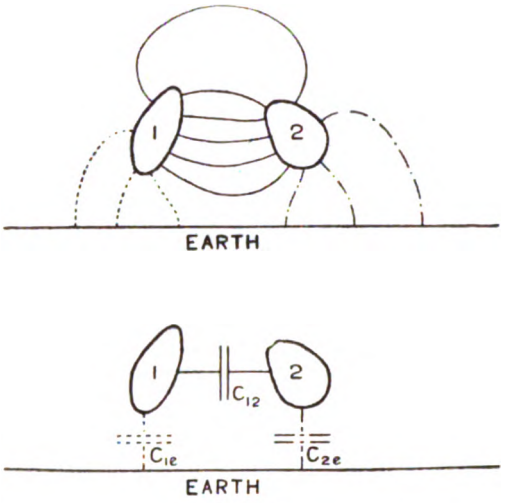

FIG. 1.

A condenser may be regarded as two conductors surrounded by a dielectric medium. Denoting the conductors by 1 and 2 (Fig. 1), MAXWELL showed that the charges on these are governed by the equations

$$E_1 = q_{11} V_1 + q_{12} V_2 \quad \ldots\ldots\ldots\ldots (1)$$

$$E_2 = q_{21} V_1 + q_{22} V_2. \quad \ldots\ldots\ldots\ldots (2)$$

Here E_1, E_2 are the charges, and V_1, V_2 the potentials of the conductors 1 and 2 respectively. The coefficients q_{11}, q_{22} are the coefficients of capacity, and q_{12}, q_{21} are the coefficients of induction. As is well known, the coefficients q_{11}, q_{22} are positive, while q_{12} and q_{21} are both negative. Also $q_{12} = q_{21}$.

Now suppose 2 is earthed and 1 is raised to potential V_1. Then by the above equations the charge on 1 is $q_{11}V_1$, and that on 2 is $q_{12}V_1$. Thus we may say that $q_{11}V_1$ tubes of force emanate from 1, and of these $-q_{12}V_1$ end on 2, and the rest, $(q_{11} + q_{12})V_1$, end on the walls of the room, *i.e.*, proceed to earth. Similarly, if 2 is raised to potential V_2 and 1 is earthed, the charge on 2 is $q_{22}V_2$ and that on 1 is $q_{21}V_2$. Thus of the $q_{22}V_2$ tubes of force which start from 2, $-q_{21}V_2$ end on 1, and the rest, $(q_{21} + q_{22})V_2$, go to earth. When 1 is at potential V_1 and 2 at potential V_2 the equations of the charges may be written (remembering that $q_{21} = q_{12}$)

$$E_1 = (q_{11} + q_{12}) V_1 - q_{12} (V_1 - V_2), \quad \ldots\ldots\ldots (3)$$

$$E_2 = (q_{22} + q_{12}) V_2 + q_{12} (V_1 - V_2), \quad \ldots\ldots\ldots (4)$$

and we may interpret them as follows: $(q_{11} + q_{12}) V_1$ is the number of tubes of force

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proceeding from 1 to earth, $(q_{22} + q_{12})\,V_2$ is the number proceeding from 2 to earth, while $-q_{12}\,(V_1 - V_2)$ is the number starting from 1 and ending on 2. Thus $(q_{11} + q_{12}) = C_{1e}$ may be regarded as the capacity of 1 with respect to earth or the "earth capacity" of 1, $(q_{22} + q_{12}) = C_{2e}$ is the earth capacity of 2, and $-q_{12} = C_{12}$ may be regarded as the capacity between 1 and 2 or the "intercapacity" of 1 and 2. These quantities are represented diagrammatically in Fig. 1. The equations now become

$$E_1 = C_{1e}\,V_1 + C_{12}\,(V_1 - V_2) \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$E_2 = C_{2e}\,V_2 - C_{12}\,(V_1 - V_2). \quad \ldots\ldots\ldots\ldots \quad (6)$$

The coefficients C_{12}, C_{1e}, C_{2e} depend only on the size, shape, and position of the conductors and the nature of the surrounding medium, and not at all on the potentials or charges. They cannot be negative, and C_{1e} is always less than q_{11}.

The question arises as to what is to be understood by "the capacity of the condenser." This is sometimes defined as follows: Let one conductor be given a charge $+q$ and the other a charge $-q$, then if V is the difference of potential between them, the capacity of the condenser is the ratio q/V. It is easy to see that in this case, since $E_1 = -E_2 = q$, the potentials of the conductors must satisfy the relation $C_{1e}V_1 = -C_{2e}V_2$, and that

$$\text{capacity of condenser} = \frac{q}{V_1 - V_2} = C_{12} + \frac{C_{1e}C_{2e}}{C_{1e} + C_{2e}}.$$

An alternative definition is: Let one conductor be earth-connected; then the charge which must be given to the other conductor to raise it to unit potential is the capacity of the condenser. In this case, if 2 is the earth-connected conductor, we have by (5)

$$\text{capacity of condenser} = C_{12} + C_{1e},$$

and, if 1 is the earth-connected conductor,

$$\text{capacity of condenser} = C_{12} + C_{2e}.$$

Now it is obviously desirable that a condenser should be capable of being regarded as a piece of apparatus with a definite constant capacity. On the above definitions, however, the capacity of the condenser is a function of C_{1e} and C_{2e}, and these vary as the condenser is moved from one place to another. From this point of view it is better to define the capacity of the condenser as the quantity of negative charge induced on either plate of the condenser (*i.e.*, either conductor) when this plate is earth-connected and the other plate is raised to unit potential. According to this definition,

$$\text{capacity of condenser} = C_{12}.$$

The differences between the values of the capacity determined by these three definitions are, of course, only of importance in the case of small condensers, say 0.001 microfarad

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and less. Such condensers should always be completely enclosed in a metallic screen, which may be used in two ways.

(*a*) The screen is connected to one plate of the condenser, say to conductor 1, *i.e.*, conductor 2 is now of such a form that it completely surrounds 1. In this case, as is well known, C_{1e} is reduced to zero, C_{2e} has become larger and thus possibly of greater importance, but the advantage of having practically eliminated C_{1e} outweighs this disadvantage.

(*b*) The screen is earth-connected, but not connected to either plate. The effect of this is to make both C_{1e} and C_{2e} quite definite and independent of the position of the apparatus in the laboratory, although neither of them is zero.

In practice small condensers are almost exclusively used with alternating current, and not simply with static charges. What we are concerned with is the relation between the current flowing in or out of the two conductors and the potential difference between them. The general relations between the potentials and currents of the system are obtained simply by differentiating (5) and (6) with respect to time. We have

$$i_1 = C_{1e}\dot{V}_{1e} + C_{12}\dot{V}_{12}, \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$i_2 = C_{2e}\dot{V}_{2e} - C_{12}\dot{V}_{12}, \quad \ldots\ldots\ldots\ldots \quad (8)$$

where

i_1 = current flowing into plate 1,

i_2 = current flowing into plate 2,

$\dot{V}_{12} = \frac{d}{dt}(V_1 - V_2)$, $\dot{V}_{1e} = \frac{dV_1}{dt}$, and so on.

It is customary to think of the current as flowing into one plate of a condenser and out of the other. Thus we may write:

current flowing into plate 1 $= C_{1e}\dot{V}_{1e} + C_{12}\dot{V}_{12}$,

,, ,, out of ,, 2 $= -C_{2e}\dot{V}_{2e} + C_{12}\dot{V}_{12}$.

Thus, in general, the current flowing into one plate is not equal to that flowing out of the other. The effective value of the capacity of the condenser in any circuit is the ratio of the current flowing into one of the plates to the rate of change of the potential difference between the two.

Thus

$$\text{effective capacity of condenser} = \frac{i_1}{\dot{V}_{12}} = C_{12} + C_{1e}\frac{\dot{V}_{1e}}{\dot{V}_{12}},$$

or

$$= \frac{-i_2}{\dot{V}_{12}} = C_{12} - C_{2e}\frac{\dot{V}_{2e}}{\dot{V}_{12}}.$$

Thus, in general, the value depends on the potential distribution, in addition to which

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different values are obtained depending on which of the currents i_1 or i_2 flows through the rest of the measuring apparatus, *i.e.*, which is the effective current. In three cases the effective capacity becomes quite definite and independent of the voltage:

(*a*) When 1 is always kept at earth potential, *i.e.*, $V_{1e} = 0$, and i_1 is the effective current.

(*b*) When 2 completely surrounds 1, *i.e.*, 2 is connected to shield ($C_{1e} = 0$), and i_1 is the effective current.

(*c*) When the currents i_1 and $-i_2$ are equal.

In cases (*a*) and (*b*) the effective capacity becomes C_{12}. Devices for eliminating the "earth capacity effects" in alternating current bridges are arrangements for satisfying one or other of these conditions. Consider, for example, the measurement of the capacity of a small condenser on the Carey Foster bridge (Fig. 2). Here it is obvious

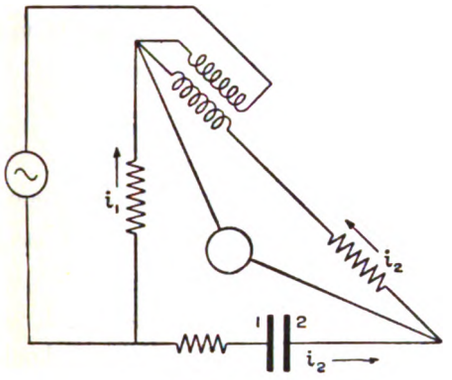

FIG. 2.

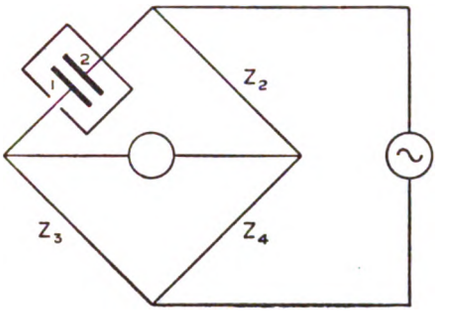

FIG. 3.

that the effective current is that flowing out of plate 2 and not that flowing into plate 1. WAGNER'S earthing device maintains the potential of 2 permanently zero and thus condition (*a*) is satisfied, whether the condenser is shielded or not.

Again, consider the capacity bridge shown in Fig. 3. Z_2, Z_3, and Z_4 are standard impedances of some form or other. Here the effective current is evidently that flowing out of 1, *i.e.*, that flowing through Z_3. If then 2 is made to surround 1 completely, the effective capacity is again C_{12}, condition (*b*) being satisfied. In actual practice the screening is never perfect, and it is advisable to make the potential of 1 as nearly zero as possible so as to make the residual error as small as possible.

The condition that the currents i_1 and $-i_2$ are equal is by (7) and (8)

$$C_{1e}\dot{V}_{1e} + C_{2e}\dot{V}_{2e} = 0,$$

from which, remembering that $\dot{V}_{12} = \dot{V}_{1e} - \dot{V}_{2e}$, we obtain

$$\dot{V}_{1e} = \frac{C_{2e}}{C_{1e} + C_{2e}}\dot{V}_{12} \quad \text{and} \quad \dot{V}_{2e} = \frac{-C_{1e}}{C_{1e} + C_{2e}}\dot{V}_{12}.$$

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Thus under these conditions the effective capacity is

$$C_{12} + \frac{C_{1e}\,C_{2e}}{C_{1e} + C_{2e}},$$

which is obviously the value for the combination of capacities shown in Fig. 1 obtained by applying the usual formulæ for condensers in series and in parallel.

Consider now an actual laboratory condenser. It usually consists of two sets of plates a, b, separated by some dielectric and surrounded by a metallic shield s (Fig. 4). It may be used in two ways:

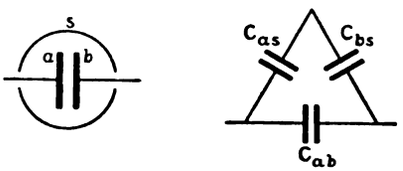

FIG. 4.

(1) The shield is connected to one set of plates, and the arrangement consists simply of two conductors, 1 and 2, say, arranged so that 2 completely surrounds 1. If the capacity of such a condenser is to be regarded as a constant of the instrument independent of its position and the voltage applied, then it must be defined as C_{12} in accordance with what has been stated above. It is easy to see that

$$C_{12} = C_{ab} + C_{as}$$

when s is connected to b. When a condenser is stated to be calibrated with screen connected to plate b, say, it should be understood that this is the quantity which has been measured.

(2) If the shield is not connected to one plate it should be earth-connected, and in order to be consistent with the above, the capacity under these conditions should be defined as C_{ab}, and when a condenser is stated to be calibrated with shield earthed, this is the quantity which should be measured.

Experience of condensers on the market shows, however, that when a condenser is stated to be "calibrated with screen earthed" the value referred to is usually $C_{ab} + \frac{C_{as}C_{bs}}{C_{as}+C_{bs}}$, *i.e.*, the effective value when condition (*c*) is satisfied. It must be recognised that the effective capacity is only given by this expression when, in addition to the screen being earthed, the potential distribution in the circuit is such that the potentials of the plates a and b satisfy the relation $C_{as}\dot{V}_{as} + C_{bs}\dot{V}_{bs} = 0$. In practice this condition is seldom satisfied.

If the shield is insulated from both plates and not earthed, then the conditions are more complicated. The shield must be considered as a separate conductor (previously it was only part of the "earth"). The general equations for the current flowing into these conductors are obtained from MAXWELL'S equations exactly as was done for two conductors. We have:

$$\begin{aligned} i_1 &= C_{1e}\dot{V}_1 + C_{12}\dot{V}_{12} + C_{13}\dot{V}_{13} \\ i_2 &= C_{2e}\dot{V}_2 - C_{12}\dot{V}_{12} + C_{23}\dot{V}_{23} \\ i_3 &= C_{3e}\dot{V}_3 - C_{23}\dot{V}_{23} - C_{13}\dot{V}_{13}. \end{aligned}$$

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If conductor 3 represents the shield and no current is led into it, we have

$$i_3 = 0.$$

Also, if conductor 3 completely surrounds conductors 1 and 2,

$$C_{2e} = C_{1e} = 0.$$

Inserting these conditions and taking for the effective capacity the ratio $i_1/\dot{V}_{12}$, we find

$$\text{effective capacity} = C_{12} + \frac{C_{13}\,C_{23}}{C_{3e} + C_{31} + C_{32}} + \frac{C_{3e}\,C_{31}}{C_{32} + C_{31} + C_{3e}} \cdot \frac{\dot{V}_1}{\dot{V}_1 - \dot{V}_2},$$

i.e., the effective capacity depends on the potentials and also on C_{3e}, *i.e.*, the earth capacity of the shield, which may be large, and will vary as the condenser is moved about or as the observer moves about the condenser.

The conclusion to be drawn is that the effective capacity of a small condenser placed in an alternating current circuit depends on the potential distribution imposed on it by the circuit. In calibrating a standard condenser it is essential to make sure that the conditions are such that the effective capacity measured is a function of the geometrical properties of the condenser alone (and the nature of the dielectric, of course) and it is necessary to specify precisely what combination of the three constants of the condenser C_{ab}, C_{as}, C_{bs} is actually measured.

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VIII.

THE INPUT IMPEDANCES OF THERMIONIC VALVES AT LOW FREQUENCIES

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ABSTRACT.

It is shown that accurate measurements of input admittance (or of input impedance) under various conditions can be made by means of the Schering capacity bridge. The input circuit is regarded as being equivalent to a condenser with a definite phase angle ϕ, or "loss angle" $\delta = 90° - \phi$, and the results are expressed by stating the effective capacity and value of tan δ for each set of experimental conditions.

A series of measurements made on an R valve is recorded, and it is shown that the results are in good agreement with the theoretical investigations of MILLER and NICHOLS. The theoretical investigation has been extended to allow for the effect of dielectric losses in the valve, since these were found by experiment to be rather large, and to have an appreciable effect on the capacity and phase angle of the input circuit.

It is shown that the effective capacity may vary from about 10 $\mu\mu$F to 100 $\mu\mu$F for an R valve, and the phase angle may vary from about 80° leading to 126° leading, depending mainly on the load in the anode circuit. Values of phase angle greater than 90° correspond to a negative resistance, or negative power factor, and occur when the load in the anode circuit is inductive. The variations of input capacity and phase angle with filament voltage, anode voltage, input voltage, and frequency are also investigated.

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THE INPUT IMPEDANCES OF THERMIONIC VALVES AT LOW FREQUENCIES.

By L. HARTSHORN, B.Sc., A.R.C.S., D.I.C., Senior Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 39, 1927.]

Introductory.

The input impedance of a thermionic valve has been investigated theoretically by Miller* and Nichols,† who showed :—

(1) That, owing to the effect of the capacity between the grid and anode, the input impedance varies with the load in the anode circuit.

(2) That the input impedance cannot in general be considered as a pure capacity, even when a large negative grid bias is used. It may be considered as a capacity in series with a resistance, or as a condenser with a power factor, which is often quite large, and may be negative when the anode circuit contains an inductive load.

Miller made measurements of the input capacity at audio frequencies when the load in the anode circuit consisted of a pure resistance, and the results showed that his formula for the input capacity was correct. He was not, however, able to check the formula for the equivalent series resistance, since he found that this was largely due to dielectric losses in the glass of the valve in the cases he was investigating, and these losses had been neglected by him in obtaining the formulæ.

Morecroft‡ has made some measurements of input capacity and conductance at a frequency of 500 kilocycles, with both pure resistance and inductive loads in the anode circuit, though he did not attempt a complete analysis of the circuits.

In working with the Schering bridge at audio frequencies, it was found possible to make measurements of the impedances of valves with considerable precision, so that it seemed to be worth while to attempt to verify the theoretical equations for input impedance for the important case of an inductive load in the anode circuit. Since the electrode capacities were found to be by no means pure, the theoretical treatment was first extended to include the effects of dielectric losses in the valve.

* Miller, 'Bull. Bureau of Standards,' Vol. 15, Bull. 351, p. 367, 1919.

† Nichols, 'Phys. Rev.,' Vol. 13, p. 411, 1919.

‡ Morecroft, 'Proc. Inst. Radio Eng.,' Vol. 8, p. 239, 1920; 'The Principles of Radio Communication,' 1st Ed., p. 428 (Wiley, New York) 1921.

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Theoretical Treatment.

Following NICHOLS, the valve circuit may be represented by Fig. 1. Here F, G and P represent the filament, grid and plate respectively. The inter-electrode capacities are C_1, C_2 and C_3, and these are not considered to be pure, but to possess loss angles δ_1, δ_2 and δ_3 respectively, or, alternatively, there is associated with C_1 a conductance

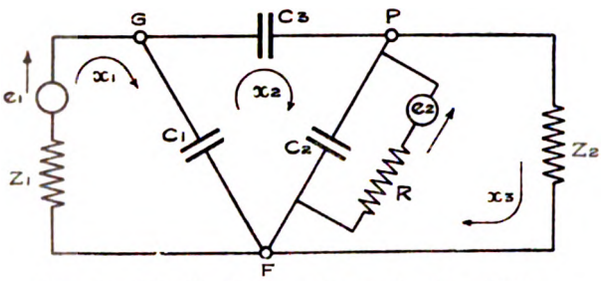

FIG. 1.—The Equivalent Network of the Valve.

G_1, such that $G_1/C_1\omega = \tan\delta_1$, and so on. Z_1 represents the external impedance in the grid circuit, and Z_2 is that in the anode circuit. An E.M.F. e_1 is introduced into the grid circuit, and as a consequence we have the mesh currents x_1, x_2, x_3 in the network. The P.D. between the grid and filament is evidently $(x_1 - x_2)/\alpha_1 = \delta v$, where α_1 represents the conductance operator of the imperfect condenser C_1, *i.e.*, $\alpha_1 = G_1 + jC_1\omega$.

Therefore, we must consider an E.M.F. $e_2 = -\mu\delta v = \mu(x_1 - x_2)/\alpha_1$ to be introduced into the plate circuit, μ being the magnification constant of the valve. The direction of the E.M.F. $\mu\delta v$ must be such as to increase the plate current when δv makes the grid more positive. At low frequencies, the capacity C_2 may be neglected, since it is shunted by the resistance R, and its value will only be a few micro-microfarads. KIRCHOFF'S equations for the network may now be represented thus :—

x_1	x_2	x_3	
$Z_1 + \frac{1}{\alpha_1}$	$-\frac{1}{\alpha_1}$	0	e_1
$-\frac{1+\mu}{\alpha_1}$	$\frac{1+\mu}{\alpha_1} + \frac{1}{\alpha_3} + R$	$-R$	0
$\frac{\mu}{\alpha_1}$	$-R - \frac{\mu}{\alpha_1}$	$R + Z_2$	0

Here α_3 is the conductance operator of the imperfect condenser C_3, *i.e.*, $\alpha_3 = G_3 + j\omega C_3$. Let D represent the operational determinant of this system of equations, and let D_{11} represent the minor of the element (1, 1), then

$$\frac{x_1}{e_1 D_{11}} = \frac{1}{D}.$$

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Therefore the input impedance Z_g is given by

$$Z_g = \frac{e_1}{x_1} = \frac{D}{D_{11}} \qquad \text{with } Z_1 = 0.$$

This reduces to

$$Z_g = \frac{1}{\alpha_1}\left[\frac{1 + W\alpha_3}{1 + W\alpha_3 + \frac{\alpha_3}{\alpha_1}\left(1 + \mu - \mu\frac{W}{Z_2}\right)}\right]$$

where

$$W = \frac{RZ_2}{R + Z_2}.$$

The input impedance will be measured as a capacity, and therefore it will be more convenient to consider the admittance

$$\frac{1}{Z_g} = \frac{\alpha_1\left[1 + W\alpha_3 + \frac{\alpha_3}{\alpha_1}\left(1 + \mu - \mu\frac{W}{Z_2}\right)\right]}{1 + W\alpha_3} \quad \ldots\ldots\ldots \quad (1)$$

At telephonic frequencies α_3 is of the order 10^{-7}; and therefore since W is of the order 10^4, the term $W\alpha_3$ may be neglected in comparison with unity as a first approximation. Thus

$$\frac{1}{Z_g} = \alpha_1\left[1 + \frac{\alpha_3}{\alpha_1}\left(1 + \mu - \mu\frac{W}{Z_2}\right)\right] \quad \text{approx.}$$

$$= \alpha_1 + \alpha_3\left(1 + \mu - \mu\frac{W}{Z_2}\right) \quad \text{approx.}$$

Substituting in this expression

$$\alpha_1 = G_1 + jC_1\omega,$$
$$\alpha_3 = G_3 + jC_3\omega,$$

and

$$\frac{W}{Z_2} = \frac{R}{R + Z_2} = \frac{R}{R + R_2 + jX_2} = \frac{R(R + R_2)}{(R + R_2)^2 + X_2^2} - \frac{jX_2R}{(R + R_2)^2 + X_2^2}$$

(where R_2 and X_2 are the external resistance and reactance in the plate circuit), we find

$$\frac{1}{Z_g} = G_1 + G_3\left[1 + \mu - \frac{\mu R(R + R_2)}{(R + R_2)^2 + X_2^2}\right] - \frac{\mu RC_3\omega X_2}{(R + R_2)^2 + X_2^2}$$
$$+ j\omega\left\{C_1 + C_3\left[1 + \mu - \frac{\mu R(R + R_2)}{(R + R_2)^2 + X_2^2} + \frac{\mu RC_3X_2\tan\delta_3}{(R + R_2)^2 + X_2^2}\right]\right\}.$$

Separating this into the capacity and conductance components, we see that the input capacity C_g is given by

$$C_g = C_1 + C_3 + \mu C_3 - \frac{\mu RC_3[(R + R_2) - X_2\tan\delta_3]}{(R + R_2)^2 + X_2^2} \quad \ldots\ldots\ldots \quad (2)$$

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The effective conductance is similarly given by

$$G_g = \left[G_1 + G_3 + \mu G_3 - \frac{\mu R G_3 (R + R_2)}{(R + R_2)^2 + X_2^2} \right] - \frac{\mu R C_3 \omega X_2}{(R + R_2)^2 + X_2^2} \quad . \quad . \quad . \quad . \quad (3)$$

The phase angle ϕ_g, or the loss angle δ_g of the input impedance, may now be calculated from

$$\tan \delta_g = \cot \phi_g = \frac{G_g}{C_g \omega}.$$

The following points may be noted :—

(1) When the load consists of a pure resistance, the formula for the input capacity becomes

$$C_g = C_1 + C_3 + \mu C_3 - \frac{\mu R C_3}{R + R_2} = C_1 + C_3 + \frac{\mu R_2 C_3}{R + R_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the formula given by MILLER for this case. The loss angle of the valve capacities does not appear in this expression, which explains the fact that MILLER was able to verify the equation although he neglected these power losses.

The conductance in this case is given by

$$G_g = G_1 + G_3 + \mu G_3 - \frac{\mu R G_3}{R + R_2} = G_1 + G_3 + \frac{\mu R_2 G_3}{R + R_2},$$

and, therefore, the phase angle is given by

$$\cot \phi_g = \tan \delta_g = \frac{G_1 + G_3 + \mu R_2 \dfrac{G_3}{R + R_2}}{\omega \left(C_1 + C_3 + \mu R_2 \dfrac{C_3}{R + R_2} \right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Evidently, if $\tan \delta_1 = \tan \delta_3$, then $\tan \delta_g = \tan \delta_1$, and does not vary with the load. In practice $\tan \delta_3$ is often less than $\tan \delta_1$, and consequently $\tan \delta_g$ decreases slightly as the load is increased.

(2) When the load in the anode circuit is reactive, the input capacity is a function of this reactance, and is also affected by the dielectric losses associated with the grid-anode capacity ($\tan \delta_3$). Also when the reactance is positive (the load is inductive), the input conductance contains a negative term. This is often much larger than the positive term, which depends almost entirely on the dielectric losses in the valve as explained above. If these losses are such that $\tan \delta_1 = \tan \delta_3$, then to a first approximation we may write the expression for the phase angle of the input circuit as

$$\cot \phi_g = \tan \delta_g = \tan \delta_3 - \frac{\mu R C_3 X_2}{C_g [(R + R_2)^2 + X_2^2]}.$$

This equation, and also the capacity equation, have been verified by measurements made on an R valve at audio frequencies.

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Experimental Investigation.

As a preliminary, the electrode capacities and power factors of the valve in its holder were determined. The Schering bridge* was used for this purpose (see Fig. 2). The electrodes of which the mutual capacity was to be measured were connected in parallel with the standard variable air condenser C_1. The third electrode was connected to earth. Balance was obtained with the valve so connected, by adjusting C_1 and C_4 (or C_3). The electrode which was connected to the point A of the bridge was then disconnected from the lead at X, and then the bridge re-balanced by varying C_1 and C_4

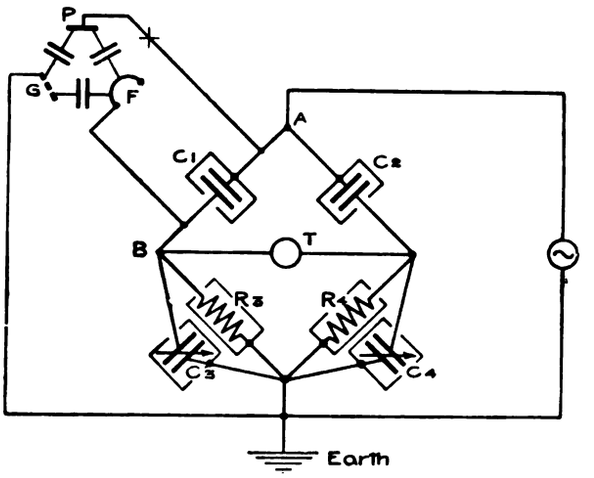

FIG. 2. Bridge for the Measurement of Electrode Capacities.

only. The difference between the two readings of C_1 gives the capacity between the two electrodes connected to A and B. The capacities between these electrodes and the third electrode, which is earthed, do not affect the reading of C_1 appreciably, since one of these capacities is merely a shunt across the whole bridge, and the other is thrown in parallel with C_3, and thus only affects C_4. The change in the reading of C_4 on disconnecting at X and re-balancing is due entirely to the change in phase angle of the impedance AB on disconnecting the valve, *i.e.*, it depends only on the phase angle of the valve impedance. (If disconnexion were made on the B side of the arm AB, the value of C_4 would be affected, due to changes in earth capacities altering the value of C_3. Since the earth capacities at A form merely a shunt across the whole bridge, they may be altered without affecting C_4.) If now C_4' is the reading of C_4 when the valve is connected to the bridge, and C_4'' when it is disconnected at X, then the power factor of the valve capacity which is being measured is given by

$$\tan \delta = \frac{C_1}{C_v} \cdot \omega \cdot R_4 \cdot (C_4' - C_4''),$$

where

C_1 = value of C_1 when valve is disconnected.

C_v = valve capacity, measured as explained above.

$\omega = 2\pi \times$ frequency.

* See HARTSHORN, 'Beama,' Vol. 13, p. 92, 1923; or 'N.P.L. Collected Researches,' Vol. 21, Paper 7, p. 116 ('Proc. Phys. Soc. Lond.,' Vol. 36, p. 401, 1924).

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In this way the capacities and power factors of the individual valve capacities were all measured.

The capacities and power factors of the valves may be also measured in pairs, in the manner adopted by MILLER, and the individual capacities and power factors then obtained by calculation. The method here described is, however, much more direct, and also takes much less time.

An R valve was tested by both methods and the results were found to agree to within 0.05 $\mu\mu$F. In general, the uncertainties due to the effect of leads will be greater than this. The results obtained for the R valve used are given in Table I. The values of power factor are seen to be very considerable. Probably they are higher than the average for an R valve,* but the fact that such power factors are not uncommon should be recognised.

TABLE I.—Capacity and Power Factor of Valve R10 in Holder.

Frequency, 1,000 cycles per sec.

Electrodes.	Capacity, $\mu\mu$F.	Tan δ.
Filament-plate	$8\cdot_6$	0·24
Filament-grid	$8\cdot_7$	0·20
Grid-plate	$3\cdot_3$	0·13

These high power factors are probably due to absorption effects in the glass, and consequently they are accompanied by changes in the capacity with frequency. The results given in Table II illustrate this.

TABLE II.—Change of Capacity with Frequency for Valve R10.

Frequency, cycles per sec.	Capacity, $\mu\mu$F.	Power factor.
500	$9\cdot_9$	$0\cdot2_0$
1,000	$8\cdot_7$	$0\cdot2_0$
2,000	$7\cdot_6$	$0\cdot2_0$
3,000	$7\cdot_0$	$0\cdot2_2$

The other capacities of this valve, and also those of other valves of the same make, behaved in much the same way.

The input impedance was also measured on the Schering bridge, the connexions being shown in Fig. 3 which is self-explanatory. The impedance, which may be regarded as an imperfect condenser, was measured in exactly the same way as the valve capacities and power factors, disconnexion being made at X (Fig. 3). The calculation of results is exactly the same as before. The case of negative power factor presents no difficulty; it merely means that, when the disconnexion is made, the reading of C_4 increases instead of decreasing, and thus a negative sign appears when the usual formula is applied. The physical significance of the negative power factor is as follows. The input circuit

* *Cf.* the typical values given in a paper by HARTSHORN and JONES, 'Experimental Wireless,' Vol. 2, p. 263, 1925.

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behaves like a condenser in which the energy stored when the voltage is increasing is less than that given out when it is decreasing, *i.e.*, there is a net gain of energy, this energy being, of course, received from the anode circuit by way of the grid-anode capacity. The current in the input circuit leads the potential difference between filament and grid by an angle greater than 90°, or, alternatively, the filament-grid impedance contains a negative resistance component. The valve does not, in general, oscillate, since the complete input circuit includes the bridge, which has a positive resistance in general larger than the negative resistance introduced by the valve.

The alternating voltage applied to the grid in most of these measurements was of the order 0.3 volt. The detector T consisted of an amplifier and telephone. There was no difficulty in obtaining sensitivity to 0.1 $\mu\mu$F. A grid bias of — 2 or — 4 volts was used, the anode voltage being usually of the order of 100.

Results.

(*a*) *Effect of Lighting the Filament on Valve Capacities.*—The input capacity and power factor of a valve at 1,000 cycles per sec. were measured, first with the filament current switched off, and then with various values of filament current up to the maximum. The circuit was as in Fig. 3, but there was no impedance Z_2 added in the anode circuit

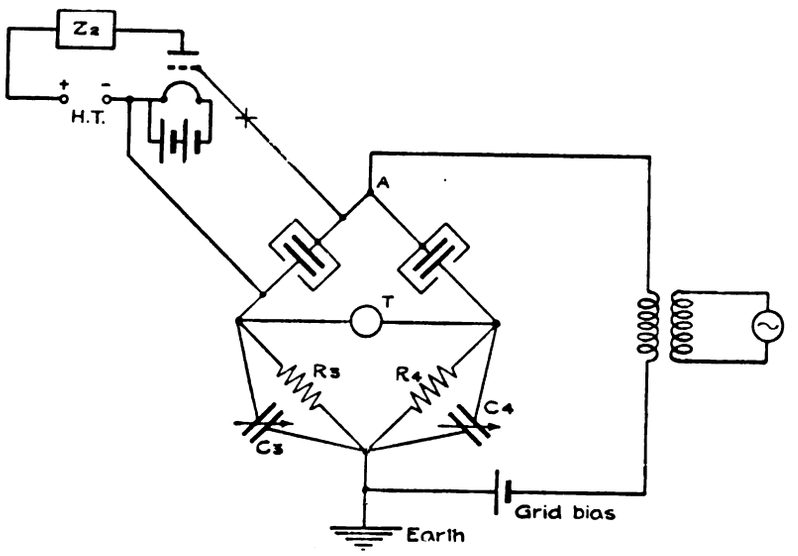

FIG. 3. Bridge for Measurement of Input Impedance.

except the resistance of the H.T. battery (about 100 ohms). The results are given in Table III.

TABLE III.—Input Capacity and Filament Current for Valve R10.

Frequency, 1,000 cycles per sec. Anode voltage, 60. Grid bias, — 2 volts.

Voltage across filament.	Input capacity, $\mu\mu$F.	Power factor (tan δ).
0·0	13·$_0$	0·21
2·7	13·$_0$	0·22
3·2	13·$_7$	0·22
4·0	14·$_6$	0·38

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It is to be noted that the capacity measured when the filament is not alight is ($C_{gf} + C_{ga}$) or ($C_1 + C_3$). When the filament is alight, the capacity measured is given by equation (4), where $R_2 = 100$ ohms, which is negligibly small since R, for a valve of this type, is about 30,000 ohms. Thus equation (4) gives for the capacity $C_g = C_1 + C_3$, *i.e.*, the capacity measured should not be altered by lighting the filament, unless the electrode capacities are affected by the presence of the electron current. Table III shows that the capacities are practically unchanged by lighting the filament. The small changes which are observed may well be due to the increased dielectric absorption effects which occur when the glass is heated. The increase in the power factor as the filament voltage is increased supports this view. Similar measurements were made on a second valve of the same type. The results were much the same, except that the increase of power factor was even greater.

(*b*) *Variation of Input Impedance with Filament Voltage, Anode Voltage and Input Voltage.*—Before attempting to verify the various formulæ developed, it is advisable to know what fluctuations in input impedance are to be expected from changes in the experimental conditions. Table IV shows the effect of variations of filament voltage on the input impedance. It is evident that, when the valve is run at its rated voltage, small variations in this voltage do not seriously alter the input impedance.

TABLE IV.—Input Impedance and Filament Voltage for R Valve.

Anode volts, 120. Grid bias, — 4 volts. Load in anode circuit, inductance coil of 5 henries. Frequency, 1,000 cycles per sec.

Filament voltage.	Input capacity, C_g, $\mu\mu$F.	Tan δ_g.
4·0	64·$_7$	—0·50
3·9	64·$_6$	—0·50
3·8	62·$_3$	—0·55
3·7	60·$_3$	—0·58

Variations of anode voltage were found to cause larger variations in the input impedance. Fig. 4 shows the results obtained with an R valve. In subsequent tests made with this valve the anode voltage was maintained at 120, since at this point the changes of input impedance due to small changes of anode voltage are not large.

As previously stated, the majority of these measurements were made with a P.D. of 0.3 volt applied to the grid circuit of the valve. In order to find out whether the input impedance varied with the applied voltage, a series of measurements was made on one circuit, using various values of input voltage. The results are shown in Table V.

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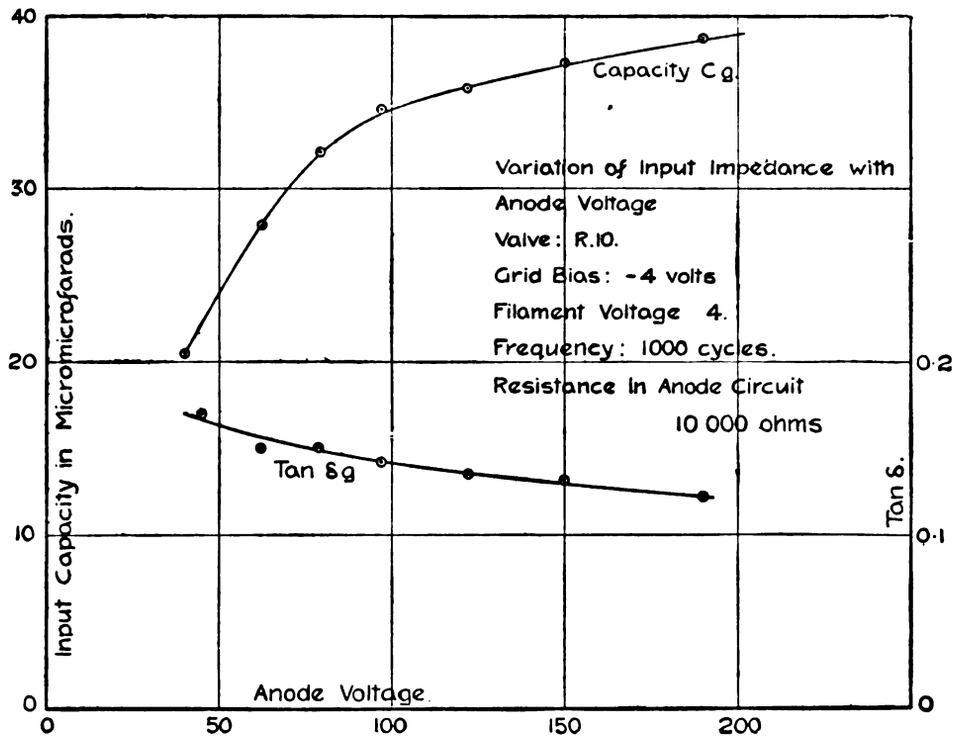

FIG. 4.

TABLE V.—Input Impedance and Input Voltage for Valve R10.

Anode voltage, 120. Grid bias, — 4 volts. Filament voltage, 4. Load in anode circuit, inductance coil of 5 henries. Frequency, 1,000 cycles per sec.

Input voltage.	Input capacity, C_g, $\mu\mu$F.	Tan δ_g.
0·33	66·$_7$	—0·49
0·16	66·$_6$	—0·49
0·11	66·$_5$	—0·49
0·03	67	—0·49

Equally constant results were obtained when the load in the anode circuit was a pure resistance. Within the experimental error, the value of the input impedance is independent of the applied voltage over the range 0.03 to 0.3 volt. The bridge and detecting arrangements used were not sufficiently sensitive to make reliable measurements with an applied P.D. less than 0.03 volt, but it seems probable that the input impedance can be regarded as completely independent of the applied voltage unless this is made very large.

(c) *Variation of Input Impedance with Load in the Anode Circuit.* (i) *Pure Resistance Load.*—Standard wire-wound non-reactive coils of resistances 10,000 to 50,000 ohms were inserted in the anode circuit and measurements of input impedance made for each value used. In each case the H.T. battery voltage was adjusted until the P.D. between

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the anode and negative filament terminal was 120 volts approximately. The results are shown in Fig. 5. The input capacity is seen to increase as the added resistance R_2 increases. This is in accordance with equation (4),

$$C_g = C_1 + C_3 + \frac{\mu R_2 C_3}{R + R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

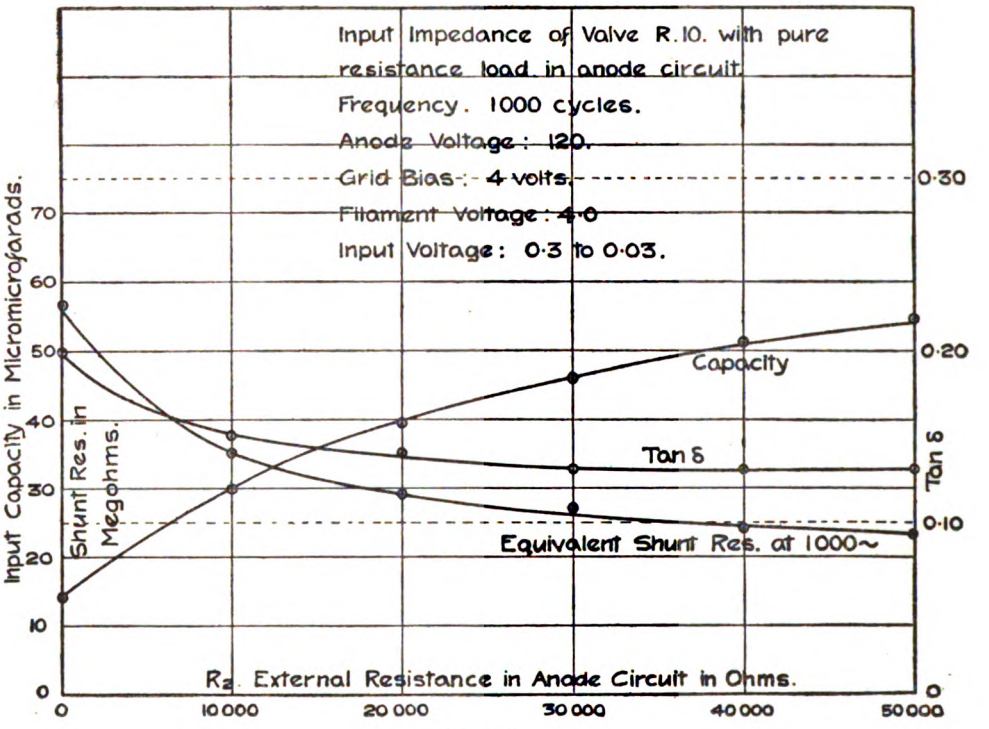

FIG. 5.

The value of C_g, when the added resistance is zero, is $C_1 + C_3$, and thus this quantity is measured directly. The capacity increment $C_g - (C_1 + C_3)$ can therefore be readily obtained for each value of R_2. Equation (4) may be written in the form :—

$$\frac{1}{C_g - (C_1 + C_3)} = \frac{R}{\mu C_3} \cdot \frac{1}{R_2} + \frac{1}{\mu C_3}.$$

Thus, if the reciprocal of $C_g - (C_1 + C_3)$ is plotted against the reciprocal of R_2, the result should be a straight line. This was found to be the case, and from the slope of this line, $R/\mu C_3$, and its intercept on the axis of ordinates, $1/\mu C_3$, the following values of the valve constants were obtained :—

$$R = 29{,}000 \text{ ohms.}$$
$$\mu C_3 = 63 \ \mu\mu F.$$

This value of the valve resistance is in good agreement with the value deduced from its characteristic curves. The value of μ for this valve was found to be 8.3, which gives a

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value of 7.6 $\mu\mu$F. for the grid-anode capacity C_3. The value for C_3 obtained by direct measurement (Table I) was 3.3 $\mu\mu$F. This represents the value for the valve in its holder, excluding any connecting leads. Evidently the effective value of C_3 when the valve is connected in circuit with other apparatus is higher than this. In the present case there is an additional 4.3 $\mu\mu$F. which probably represents the capacity between the leads to the grid and the leads to the anode, including the load Z_2 (Fig. 3). In Fig. 5 the observed points are marked with small circles. The capacity curve has been drawn through points calculated from equation (4), using the above values of the constants μC_3 and R. The agreement between the observed and calculated capacities is very close. Considering now the phase angle, the value of tan δ was found to decrease from the value 0.20, when the added resistance R_2 was zero, to a constant value of 0.13 for values of R_2 greater than 30,000 ohms. This is explained by equation (5). When R_2 is zero the value of tan δ_g is $\frac{G_1 + G_3}{\omega (C_1 + C_3)}$ —*i.e.*, it is the value for the two condensers C_1 and C_3 in parallel. This is evidently 0.20. If now R_2 is increased until it becomes comparable with R, tan δ_g approaches its limiting value $G_3/\omega C_3$, or tan δ_3, since in equation (5) the first terms in numerator and denominator then become small compared with the second and third terms. Thus when R_2 exceeds 30,000 ohms the value of tan δ_g is 0.13, which is the value given in Table I for the tan δ of the grid plate capacity C_3. This agreement is partly accidental, since the values given in Table I exclude capacities of leads. The existence of these would reduce the power factor, but the heating effect of the filament current increases the losses and so brings the power factor back more or less to its previous value.

The curve for the equivalent shunt resistance at 1,000 cycles per sec. is also shown in Fig. 5. This has been obtained by calculation from the values of C_g and tan δ_g. For this valve, with a pure resistance load, the equivalent shunt resistance at 1,000 cycles per sec. is of the order of 30 megohms, and varies with the load in much the same way as tan δ. At higher frequencies the equivalent shunt resistance would, of course, be much smaller, since it is roughly inversely proportional to the frequency.

(ii) *Load Purely Inductive.*—Standard self-inductance coils of various sizes from 0.1 to 31 henries were in turn inserted in the anode circuit, and measurements of input capacity and phase angle made for each case as before. The resistances of the coils were small compared with their reactances at 1,000 cycles per sec., so that the resistance was neglected, and the input capacity and phase angle measured, considered merely as a function of the reactance added in the anode circuit. The capacity results are shown in Fig. 6 and those for tan δ_g in Fig. 7. The variation of capacity should be given by equation (2), viz.,

$$C_g = C_1 + C_3 + \mu C_3 - \frac{\mu R C_3 [(R + R_2) - X_2 \tan \delta_3]}{(R + R_2)^2 + X_2^2}, \quad \ldots\ldots (2)$$

where X_2 represents the added reactance and R_2 the added resistance, which is there

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taken as negligible. When X_2 is zero the value of C_g given by the above equation is $C_1 + C_3$. Thus this quantity can be measured directly. In the present case

$$C_1 + C_3 = 15\ \mu\mu F.$$

When the added reactance X_2 is very large, we find from equation (2) that

$$C_g = C_1 + C_3 + \mu C_3.$$

Thus the input capacity C_g approaches this limit as the added reactance is increased indefinitely. In Fig. 6 the capacity is seen to approach a limiting value of 104 $\mu\mu$F. Hence

$$C_1 + C_3 + \mu C_3 = 104\ \mu\mu F.,$$

and therefore

$$\mu C_3 = 104 - 15 = 89\ \mu\mu F.$$

This value for μC_3 is higher than that obtained for a pure resistance load. This is doubtless due to a small increase in the component due to leads, etc., of the capacity C_3. The quantity $C_1 + C_3$ is also correspondingly larger in this case. Taking now the previously determined value of R—viz., 29,000 ohms—the values of C_g for various values of X_2 were calculated from (2). The value of tan δ_3 is 0.13 when the valve is cold (Table I). The value 0.1_5 was used in these calculations. The curve drawn in Fig. 6 represents the values of C_g obtained by calculation in this way. The points marked with small circles represent the observed values. The agreement is well within the experimental error, since slight changes in C_3 are almost inevitable when the added inductance is changed, *i.e.*, when a larger or smaller coil is added. The part played by power losses in the valve in increasing the input capacity is made clear by the dotted curve in Fig. 6. This represents the calculated values of C_g neglecting the term involving tan δ_3. The vertical distance between the dotted and continuous curves thus represents the capacity increment due to power losses.

Turning now to the phase angle ϕ_g of the input circuit, which is represented by tan δ_g [$= \cot \phi_g$] in Fig. 7, we see that when the added reactance is zero the value of tan δ_g is 0.21, *i.e.*, the input circuit behaves like a condenser with a power factor of approximately 20 per cent., *i.e.*, the current leads the voltage by an angle of 79°. As the added reactance is increased, tan δ_g rapidly decreases reaching a minimum value of -0.7_5 when $X_2 = 12{,}000$ ohms (approx.). At this point the input circuit behaves like a condenser with a negative power factor (sin δ_g) of 60 per cent.—*i.e.*, the current in the grid circuit leads the grid-filament P.D. by 126°—since $\tan^{-1}(-0.7_5) = \sin^{-1}(-0.60) = 36°$. As the value of X_2 is increased still further the value of tan δ_g increases, becoming positive again for very large added reactances. The full expression for tan δ_g obtained from (2) and (3) is rather cumbersome, and in order to simplify it a further approximation has been made. It is evident from (3) that the full expression for tan δ_g will contain two terms, the first being a function of the power losses

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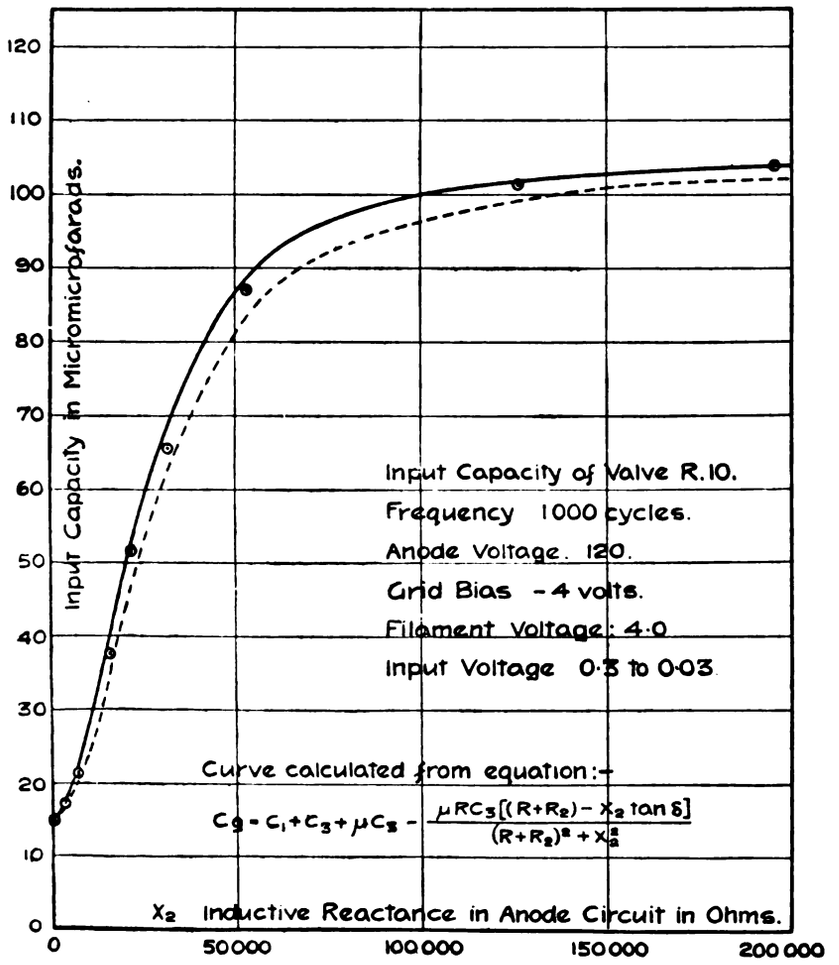

FIG. 6.

in the valve, and the second a function of the added reactance. In general, the first term will be the smaller and it varies comparatively little with the added load. It has therefore been assumed constant and has been denoted by tan δ. We then have

$$\tan \delta_g = \tan \delta - \frac{X_2 \mu R C_3}{[(R + R_2)^2 + X_2^2] C_g}. \quad \ldots\ldots\ldots (6)$$

When $X_2 = 0$ is inserted in this equation we have tan δ_g = tan δ, and therefore tan δ has been taken as 0.21, which is the measured value of tan δ_g when $X_2 = 0$. Since all the other quantities in (6) are known, the curve showing the relation between tan δ_g and X_2 can be plotted. This is the continuous curve in Fig. 7. The observed points lie on the dotted curve. The agreement between the two is as close as can be expected. The divergence is evidently due to the fact that the term tan δ is not strictly speaking constant. As a matter of fact, the value of tan δ will decrease with added load in much the same way as it does in Fig. 5, as may be seen by a comparison of the equations for the two cases. The calculated value for tan δ_g will therefore be slightly high for large

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added reactances, which explains the divergence of the two curves. We may therefore conclude that the equations given represent the input impedances of valves at low frequencies with sufficient accuracy for all practical purposes.

In this case, also, the equivalent shunt resistance curve at 1,000 cycles per sec. has been calculated from the curves of capacity and tan δ; it is shown in Fig. 7. The smallest value of the equivalent negative shunt resistance is 5 megohms at 1,000 cycles per sec.

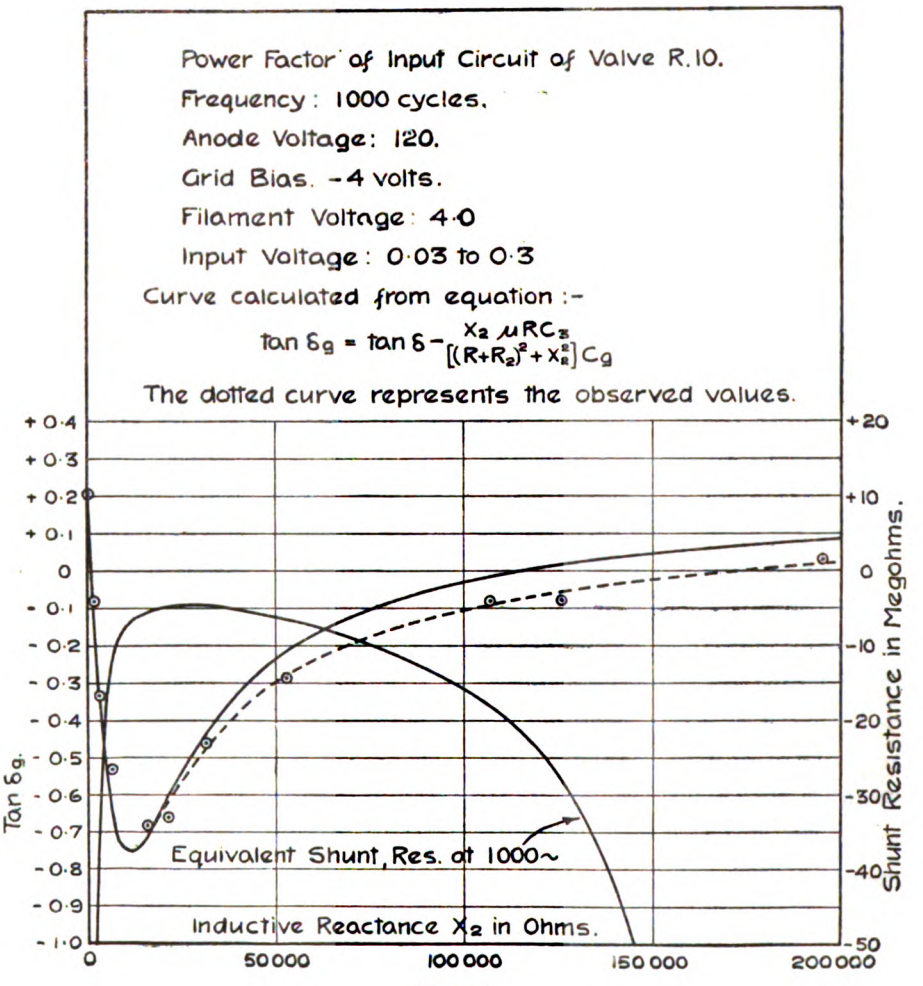

FIG. 7.

This quantity is, of course, enormously dependent on the frequency, *e.g.*, at 10,000 cycles it would be one-tenth of this amount if the value of tan δ were unchanged, and to a first approximation tan δ is independent of frequency for a given added reactance.

(iii) *Variation of Input Impedance with Frequency.*—The present investigation applies only to low frequencies (the telephonic range, say), and for this case the equations indicate that for a given added impedance in the anode circuit the input impedance is independent of frequency, provided that the valve constants μ, R, C_3, C_1 are independent of frequency. For the valve used in these experiments the values of C_1 and C_3 were found to vary with frequency (see Table II). Exactly corresponding variations were found in the input

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impedance. Fig. 8 shows curves giving the variation of input capacity with added reactance in the anode circuit for a number of different frequencies. From the curves

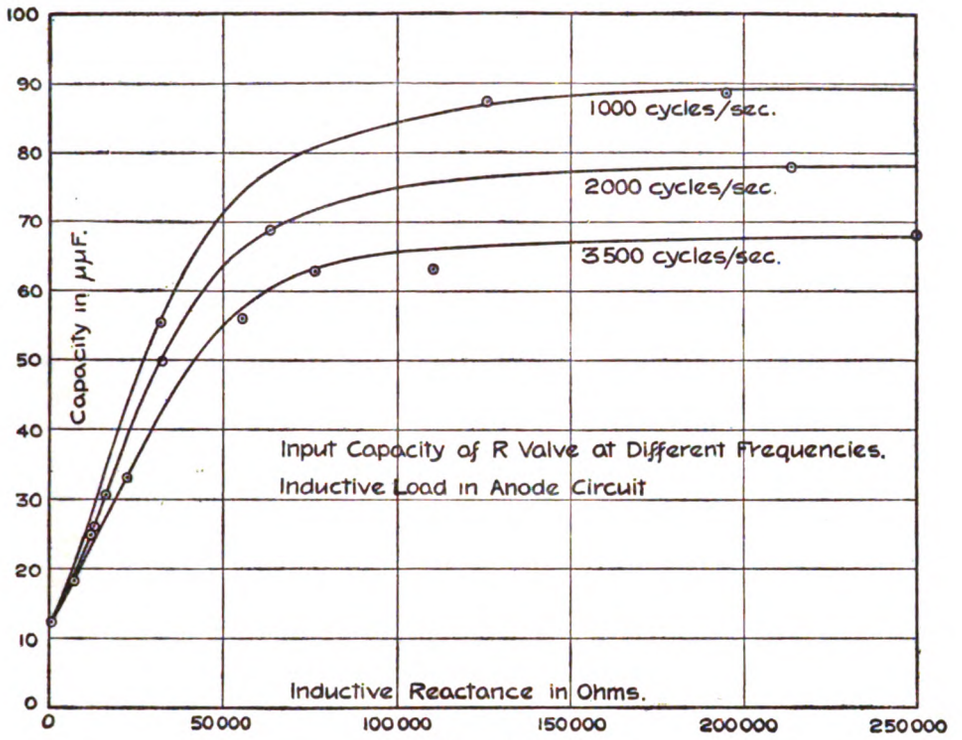

FIG. 8.

we may deduce the values of the valve constants $C_1 + C_3$, μC_3, and therefore C_3, in the manner previously indicated. The following results were obtained in this way :—

Frequency, cycles per sec.	$C_1 + C_3$.	μC_3.	C_3.
1,000	$12 \cdot_5$	76	$9 \cdot_2$
2,000	$12 \cdot_0$	66	$8 \cdot_0$
3,500	$11 \cdot_3$	58	$7 \cdot_0$

The changes are of the same order as those shown in Table II. The curves in Fig. 8 were calculated from the formula (2), using these values of $C_1 + C_3$ and μC_3 and the same values of R. The agreement is sufficient to show that the change in input capacity is due solely to the change in valve capacities. Similar results were obtained for a pure resistance load in the anode circuit.

In carrying out this investigation I have had the privilege of discussing various points with a number of my colleagues and I wish to thank them for valuable suggestions, particularly Dr. D. W. DYE, F.R.S., Mr. T. I. JONES, and Mr. F. M. COLEBROOK. My thanks are also due to Mr. D. A. OLIVER for help in making the observations.

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IX.

THE PIEZO-ELECTRIC QUARTZ RESONATOR AND ITS EQUIVALENT ELECTRICAL CIRCUIT

BY

D. W. DYE, D.Sc., A.C.G.I., F.R.S.,

PRINCIPAL ASSISTANT IN THE ELECTRICITY DEPARTMENT.

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ABSTRACT.

The quartz piezo-electric resonator is examined experimentally and theoretically with special regard to an equivalent electrical system which can represent it.

It is shown that, as theoretically predicted by BUTTERWORTH, such a resonator can be represented by an inductance, a resistance and a capacity, all in series. These are pictured as in parallel with another small condenser, and the whole is in series with a third condenser, the additional condensers representing air-gaps. The equations for the current in an oscillatory circuit, to which the resonator is attached, are developed and it is found that almost perfect agreement exists between the forms of current curve obtained theoretically and experimentally. This agreement is found to hold for longitudinal resonators of as low a frequency as 44,000 and for transverse resonators of as high a frequency as 15,000,000 cycles per second.

It is next shown how the logarithmic decrement of the resonator may be obtained from a rectified line plotted from observation on the current in the oscillatory circuit as a function of frequency width across the crevasse which pierces the summit of the resonance curve.

The methods of analysis of the equivalent mesh into its components are next developed and it is shown that this analysis can be effected by carrying out a series of observations of the current at resonance when the air-gaps are varied by known amounts, or when the effective resistance of the oscillatory electrical circuit is given different known values.

The theoretical and experimental results are found to be consistent. The value of the shunting condenser is somewhat smaller than that measured at a neighbouring frequency outside the region of resonance. A possible explanation is offered.

The effects on frequency of response of variation of air-gap are studied and the difference between prediction and observation is discussed. This difference is only a few parts in a hundred thousand. In the case of transverse resonators remarkable effects occur when the air-gap is varied through the regions where its length is an integral member of half-wavelengths of the supersonic air-waves produced by the vibration of the quartz.

The temperature coefficient of frequency of a considerable variety of resonators is examined over a range of temperatures up to 40° C. It is found that very diverse results are obtained and probable explanations are offered.

The effects of displacement of the resonator from the position of centrality are examined. These are shown to be small, but not quite negligible.

The current taken by the quartz mesh is then examined in some detail theoretically, and one or two experimental curves are given, together with a graphical method of deducing the curve of current from the constants of the quartz. A selection of theoretical curves for the conditions under which the experiments of CADY were made have been calculated. The curves are in all respects of the shape and form experimentally found by him.

A number of conclusions and suggestions terminate the paper.

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THE PIEZO-ELECTRIC QUARTZ RESONATOR AND ITS EQUIVALENT ELECTRICAL CIRCUIT.

By D. W. DYE, D.Sc., A.C.G.I., F.R.S., Principal Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 38, 1926.]

The quartz piezo-electric resonator was introduced by Cady who has published a number of valuable papers* on its properties and form. G. W. Pierce† also has contributed valuable information on the subject, showing the form, mounting and circuits of quartz oscillators whereby the resonant vibration is self-maintained by a valve in the same fashion that a tuning fork may be made self-maintaining. More recently he has published an account of some very accurate measurements of the velocity of sound at very high frequencies, using quartz oscillators as the source of sound.

Giebe‡ also has published a short paper indicating a simple and elegant means whereby the state of maximum resonant vibration may be rendered directly visible by mounting the resonator in a vacuum and observing the luminous glow between the surfaces of the quartz and the electrodes, consequent upon the large voltage gradient in the air gap at the regions where the alternating stress in the quartz is a maximum. In this manner the overtone modes of vibration may be rendered visible also. Other more or less popular accounts of the properties of quartz resonators and oscillators have from time to time appeared in the technical press. The reader is specially referred also to the classical paper by Sir William Bragg and R. E. Gibbs§ on the structure of α and β quartz.

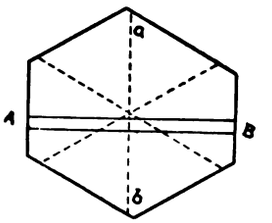

Fig. 1. Cross-section of Quartz Crystal showing how a Bar or Plate is cut.

Although, therefore, the form, construction and use of quartz resonators and oscillators may be considered fairly well known, it is, perhaps, desirable to include in the present paper the briefest description of the essentials of a piezo-electric quartz resonator.

Referring then to Fig. 1, we have a transverse section of a natural quartz crystal cut

* W. G. Cady, 'Proc. Inst. Radio Eng.,' Vol. 10, p. 83, 1922; Vol. 12, p. 805, 1924; 'J. Opt. Soc. Amer.,' Vol. 10, p. 475, 1925.

† G. W. Pierce, 'Proc. Amer. Acad. Arts Sci.,' Vol. 59, p. 82, 1923; Vol. 60, p. 271, 1925.

‡ E. Giebe and A. Scheibe, 'Zeits. f. Physik.,' Vol. 33, p. 335, 1925.

§ Sir William Bragg and R. E. Gibbs, 'Proc. Roy. Soc.,' A, Vol. 109, p. 495, 1925.

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in a plane perpendicular to the optic axis. Any such section contains three electric axes (shown dotted), at angles of 120° to each other and parallel to the rectangular bounding faces of the crystal. A bar or plate is cut from the section with its length perpendicular to one of the electric axes as shown at AB. The dimensions of the piece in the direction parallel to the optical axis of the crystal may be large or small. If it is small compared with the length AB we obtain a bar; if of the same order as AB we obtain a plate. If such a plate or bar is placed in an electric field applied in a direction perpendicular to its length, *i.e.*, parallel to the electric axis, *ab*, the bar will experience two strains, one in the direction of its length and the other perpendicular thereto, *i.e.*, along the direction of the applied field. These two strains are of opposite sign, so that if, for example, the bar becomes longer in the direction AB it becomes thinner in the direction *ab*. The two strains are such that the volume remains constant. The strain across the bar has been called the longitudinal effect because it is along the direction of the applied field, although transverse to the length of the bar.

It is clear that if an alternating electric field is applied to the bar it will be subjected to alternating stresses and strains both along its length and transversely thereto. If the frequency of the applied field is adjusted to coincide with that of one of the free modes of vibration which can be maintained by such alternating stresses, a large resonant vibration will occur. The amplitude of this vibration and its rate of change with frequency will depend upon the mechanical damping of the quartz and upon its piezo-electric reaction on the applied field.

It is the very small damping coefficient and the permanence of form and constitution of natural quartz that render it specially valuable as a radio-frequency standard when suitably cut and mounted.

It must be kept in mind in thinking of the various modes of vibration that a longitudinal vibration results from the transverse piezo-electric effect and *vice versa*.

Now a theorem has been enunciated by BUTTERWORTH* relating to a dynamical system in which mechanical vibrations may be produced by the reaction of an electric system upon it. It is shown that the mechanical vibrator may be replaced by an equivalent electrical oscillatory circuit coupled to the actual electrical circuit in a manner depending upon the nature of the mechanical oscillator and the means whereby the transformation of electrical into mechanical displacement occurs. The piezo-electric resonator is a mechanical vibrator falling within this class. For such a resonator the equivalent circuit given by BUTTERWORTH—with the addition of a small capacity K_2—is as shown in Fig. 2.

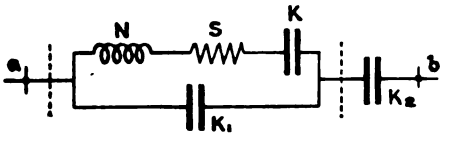

FIG. 2. The Equivalent Network representing a Resonator Mounted between Electrodes.

* S. BUTTERWORTH, 'Proc. Phys. Soc. Lond.,' Vol. 26, p. 271, 1913–14; Vol. 27, p. 410, 1914–15.

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The system consists of an inductance N,* a resistance S and a capacity K, all in series. These are shunted by another condenser K_1. Such a mesh represents a piezo-electric resonator in which the electrodes are in contact with the surface of the quartz. The constants N, K and S are related to the dynamical properties of the bar as follows—quoting from BUTTERWORTH—

$$N = \alpha/B^2 ; \quad K = B^2/\gamma ; \quad S = \beta^2/B^2,$$

where α, β and γ are the constants of inertia, damping and restoration respectively in the fundamental equation of motion

$$(\alpha D^2 + \beta D + \gamma)\, y = Y,$$

in which y is the displacement and Y is the displacing force. D is the operator d/dt. The displacing force Y is equal to Be, where e is the E.M.F. between the two surfaces of the quartz.

The constants α, β, γ are, therefore, simple physical constants of the quartz, whilst B is an electro-physical constant proportional to the piezo-electric effect. The value of B is measured by the mechanical stress produced per unit of electric field applied. It is analogous to the constant which would define the back E.M.F. produced in a wire vibrating in a magnetic field.

In the case of a bar or plate vibrating under its own internal stresses the constants of inertia, restoring force and damping are, of course, distributed. For any given mode of vibration there will, however, be equivalent concentrated inertia, damping and restoring forces which may be considered constant for that mode. If the resonator vibrates in an overtone mode such as that representing 3, 4, 5, 6, etc., segments, there will be a corresponding set of equivalent constants N, S and K, which will apply only to that mode.

It is not proposed in the present paper to examine closely the mathematical nature of the general equations given above in relation to the possible modes of vibration of the system, but to examine rather, both algebraically and experimentally, some of the properties of the equivalent electrical system of Fig. 2, and of other circuits upon which the resonance of a quartz resonator may be observed.

The resonator as usually mounted consists of a plate or bar of quartz in which the electrodes are separated from the surfaces of the bar by a small air gap. In this case we may represent the air gap by a small condenser K_2 in series, as shown in Fig. 2.

A very satisfactory method of using a resonator as a standard is that given by CADY, in which the resonator with its electrodes is connected in parallel with the condenser of an oscillatory circuit tuned approximately to the resonance frequency of the quartz. A current-measuring device in series with the inductance coil of the oscillatory circuit

* Since writing this paper a note has appeared in the ' Physical Review ' (Vol. 25, Abs. 52, p. 895, 1925) which gives identically the same circuit for the quartz resonator, but the theorem referred to was published, and the experiments described here were in progress, before this note appeared. See also footnote, p. 201.

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indicates the sudden change in current occurring at the response frequency of the quartz.

This arrangement is shown diagrammatically in Fig. 3 (*a*) and the complete electrical equivalent system is as shown in Fig. 3 (*b*). In this latter figure the piece of quartz itself is represented by N, K and S shunted by K_1. The two air gaps are represented by K_2.

A series of careful experiments is able to determine the validity or otherwise of the equivalent electrical network and will give valuable information regarding the actual values of the equivalent constants N, S and K, and show how these depend upon the

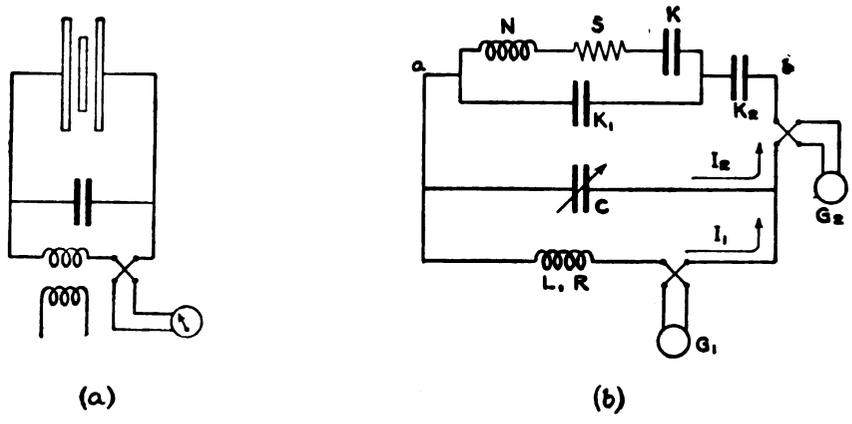

FIG. 3. Mode of observing Resonance on a Resonator and the equivalent Electrical System.

properties of the quartz, etc. It is to be anticipated, also, that much may be learned regarding the constancy of these quantities with respect to the physical condition of the specimen and its surroundings.

The experiments described herein have shown that a striking agreement may be obtained between the observed current I_1 (Fig. 3 *b*) in the oscillatory circuit as a function of the frequency of the source and the calculated current in terms of the determined constants N, S, K and K_1. Equally good agreement is also obtained between theory and experiment with respect to the current I_2 (Fig. 3 *b*) in the shunt circuit containing the resonator.

Before giving the experimental results and the comparison with those calculated, it will be found helpful, in the formation of a clear mental image of the behaviour of the resonator, to consider the properties of the equivalent electrical mesh representing the resonator itself, as an entity apart from the attached electrical circuit. We will then proceed to an examination of the properties of the complete system and will show how the analysis of the constants N, S, K and K_1 may be obtained from the experimental observations.

Referring to the mesh N, S, K, K_1 of Fig. 3 (*b*) we can find an equivalent virtual single series circuit, consisting of an effective pure resistance S_0 in series with an effective pure

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capacity K_0, to replace the mesh. The values of S_0 and K_0 will, of course, be greatly dependent upon frequency. The equivalents S_0 and K_0 will be in series with K_2 and it is this simple series circuit which is in parallel with the oscillatory circuit.

It is convenient to use a resonant frequency, $\omega_0/2\pi$, defined by $NK\omega_0^2 = 1$, and to write q for $1 - NK\omega^2$, where q is a small quantity proportional to a small change in frequency from that given by $\omega_0/2\pi$. This expression constantly occurs in what follows. The frequency $\omega_0/2\pi$ will be spoken of as the true resonant frequency, although such a definition is open to some criticism. Another quantity of great importance is the term $SK\omega$, a quantity equivalent to the power factor of the condenser K.

$SK\omega \times \pi$ is equal to the logarithmic decrement per complete period of the resonator, and will be designated δ_2. To avoid the constant repetition of π in the formulæ, $SK\omega$ will be written as ϕ_2, so that $\delta_2 = \phi_2\pi$.

Using this nomenclature it will be found that the quantities S_0 and K_0 are given by the expressions :—

$$S_0 = \frac{SK^2}{K_1^2[\phi_2^2 + (q + K/K_1)^2]} \quad \ldots\ldots\ldots\ldots \quad (a)$$

and

$$\frac{1}{K_0\omega} = \frac{\phi_2^2 + q(q + K/K_1)}{K_1\omega[\phi_2^2 + (q + K/K_1)^2]}. \quad \ldots\ldots\ldots\ldots \quad (b)$$

The magnitude which these quantities can assume is dependent mainly upon ϕ_2. This is a very small quantity—of the order 30×10^{-6} in a good specimen of quartz.

It is a property of a resonant electrical circuit that, if for various chosen values of q as defined above we plot corresponding values of S_0 and $1/K_0\omega$ as ordinate and abscissa respectively, we shall obtain an almost perfect circle. From equation (b) it will be seen also that K_0 can assume negative values for values of q lying between 0 and $-K/K_1$. This is a very important property of the quartz, and is responsible for its behaviour when placed in an air gap between two electrodes. Since it so happens that K/K_1 is a quantity of the order 1/100, the region of frequency through which $1/K_0\omega$ can be negative is small.

We have seen that the air gaps are considered as equivalent to a small series condenser K_2. If for any particular air gap represented by K_2 we choose such a value of q that $K_0 = -K_2$, the whole system between the points ab of Fig. 2 becomes equivalent to a non-inductive resistance S_1. It is clear that the maximum damping is produced upon the oscillatory circuit almost at this frequency. It is necessary to give the frequency at which this effect occurs a different name from the resonant frequency $\omega_0/2\pi$ as previously defined. It can be called the response frequency of the quartz. It differs theoretically from ω_0 by the quantity $-\frac{\omega_0 q_1}{2\pi}$, where q_1 is that value of q at which $K_0 = -K_2$. This condition is shown vectorially in Fig. 4. The curve AB shows a small part of the vector impedance circle diagram resulting from plotting $1/K_0\omega$ as abscissa against S_0 as ordinate for various values of q. Suppose the air gap is such that

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$1/K_2\omega$ is given by OQ, then, at a certain negative value of q represented by the point P on the curve, the value of $1/K_0\omega$ represented by ON will be equal and opposite to OQ. The resultant vector impedance across *ab* of Fig. 3 (*b*) is therefore OT. With a larger air gap and hence a smaller value of K_2, the negative reactance $K/K_2\omega$, shown

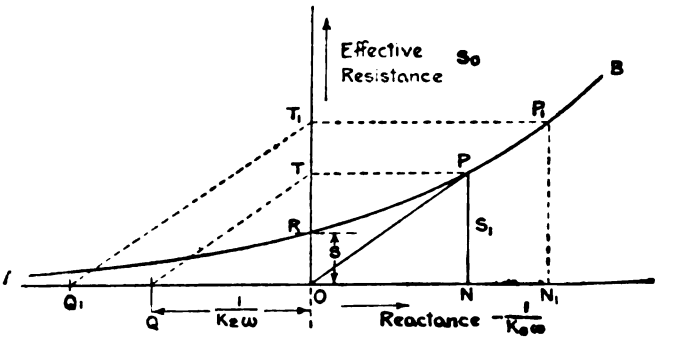

FIG. 4. Impedance Diagram of a Resonator.

as OQ_1 in Fig. 4, will be greater. The response frequency will then occur at such a value of q as is represented on the curve AB by the point P_1. This is the larger negative value of q requisite to make ON_1 equal to OQ_1. The effective resistance of the whole resonator has now the greater value OT_1, which will, therefore, not cause such a large reduction in the current in the oscillatory circuit LRC.

From these considerations it will be seen that the frequency of response is not quite equal to that which has been defined as the true resonant frequency of the quartz itself, but is a function of the air gap. When K_2 is infinite (no air gap) the response frequency is at the true resonant frequency, where $q = 0$ and the vector impedance becomes S as shown by OR in Fig. 4.

It is perhaps desirable at this stage to anticipate the experimental results somewhat by giving the actual effective values of K, S, N and K_1 experimentally measured in a typical resonator. For a bar about 7 cms. long, 0.15 cm. thick and 0.6 cm. wide, the following values have been found :—

$$K = 0.08\ \mu\mu\ F.$$

$$S = 1{,}500\ \text{ohms.}$$

$$N = 160\ \text{henries.}$$

$$K_1 = 8\ \mu\mu\ F.$$

$$\omega_0 = 275{,}000.$$

It will be seen that the reactances of K and N are very large indeed.

$\phi_2 = SK\omega$ is of the order 30×10^{-6}, giving the extremely small value of 1×10^{-4} for δ_2, the logarithmic decrement of the resonator.

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The diameter of the motional impedance circle of Fig. 4 is no less than 100 megohms. In order to show the resistance vectors OR, OT, etc., it has been necessary to magnify the ordinates by about 50, so that the circle really becomes an ellipse with a vertical major axis. Since OR represents 1,500 ohms, the major axis of the ellipse is about 300 metres long if OR is 5 mms. in the diagram.

We will now return to a consideration of the resonance curve of current in the inductance coil L of the oscillatory circuit LRC across which the resonator is shunted. It is this current which can easily be observed as a function of frequency and of those other quantities in Fig. 3 (*b*) which can be varied. By deductions made from the observations we can arrive at a complete analysis of the constants K, S, N, and K_1.

The case under investigation is as shown in Fig. 5. The resonator is the mesh $NSKK_1$, with the air gaps between the surfaces of the quartz and the electrodes represented by K_2. This system has flowing through it a current of which the instantaneous value is represented by i_2. It is shunted across the oscillatory circuit CLR, carrying the current having an instantaneous value i_1. An electromotive force e_0 (assumed constant) is induced in L by very loose coupling to a source of smoothly variable frequency as indicated. A fictitious current i_3 flows in the quartz mesh itself. This cannot be measured, but is introduced for the purposes of the analysis. It has its counterpart in the physical reaction of the quartz. The root mean square value of i_1 is measured by a heater and thermo-junction connected to galvanometer G_1. I_2 is similarly measured on G_2; this measurement is difficult owing to the smallness of I_2 and of the energy circulating in the shunt system.

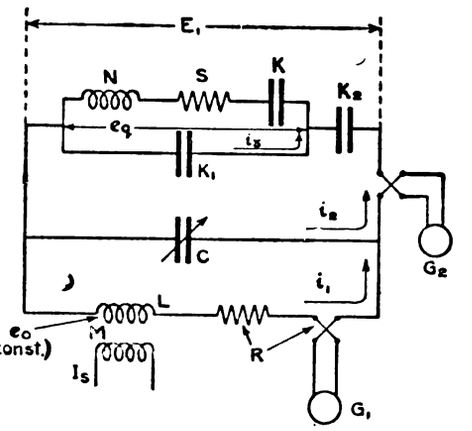

FIG. 5. Circuit Diagram or Resonator connected in parallel with an Oscillatory Circuit.

When the frequency of the source is smoothly varied through the region of response of the quartz, remarkable changes occur in the values of I_1 and I_2. Owing to the relatively large and sudden absorption of energy by the quartz at the frequency of response the current I_1 falls very rapidly to a very sharply defined minimum, and then rapidly rises again to nearly its previous value. The resonance curve of I_1, which, in the absence of the quartz, assumes a normal shape, has thus a deep and narrow crevasse. This is, of course, well known as a result of the work of CADY and others, and examples are given in his excellent paper on the subject. The current I_2 rises to a sharp maximum and suddenly falls to a small value again. The exact form of the curve I_2 and its location with respect to I_1 is, however, considerably dependent upon the conditions under which the experiments are carried out, as will be seen later.

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The equations connecting the currents and voltages represented in Fig. 5 are

$$(\mathrm{R} + \mathrm{L}\alpha)\, i_1 + \frac{1}{\mathrm{C}\alpha}(i_1 - i_2) = e_0, \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$\frac{1}{\mathrm{C}\alpha}(i_2 - i_1) + \frac{1}{\mathrm{K}_2\alpha}\, i_2 + \frac{1}{\mathrm{K}_1\alpha}(i_2 - i_3) = 0, \quad \ldots\ldots\ldots \quad (2)$$

$$(\mathrm{S} + \mathrm{N}\alpha + 1/\mathrm{K}\alpha)\, i_3 + \frac{1}{\mathrm{K}_1\alpha}(i_3 - i_2) = 0, \quad \ldots\ldots\ldots \quad (3)$$

where α is written for $j\omega$.

Both from an experimental and from a theoretical standpoint it is convenient to calculate I_1 in terms of I_0, where I_0 is the current which is obtained in the oscillatory circuit when in resonance in the absence of the quartz. We can then write $\sigma_1^2 = \mathrm{I}_1^2/\mathrm{I}_0^2$, where σ_1 is always less than unity.

By elimination of i_3 and i_2 in the above equations and by putting $\mathrm{RI}_0 = \mathrm{E}_0$, where E_0 is the r.m.s. value of e_0 in the final equation for I_1, the expression for σ_1^2 will be found to be

$$\sigma_1^2 = \frac{\phi_1^2\left[\phi_2^2 + \left(q + \dfrac{\mathrm{K}(\mathrm{C}+\mathrm{K}_2)}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2}\right)^2\right]}{\left[pq + p\dfrac{\mathrm{K}(\mathrm{C}+\mathrm{K}_2)}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2} - q\dfrac{\mathrm{K}_1\mathrm{K}_2}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2} - \dfrac{\mathrm{KK}_2}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2} - \phi_1\phi_2\right]^2} + \left[q\phi_1 + p\phi_2 + \phi_1\dfrac{\mathrm{K}(\mathrm{C}+\mathrm{K}_2)}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2} - \phi_2\dfrac{\mathrm{K}_1\mathrm{K}_2}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2}\right]^2}. \quad (4)$$

In order to render the equation more compact the following substitutions have been made :—

$$\phi_1 = \mathrm{RC}\omega, \qquad \phi_2 = \mathrm{SK}\omega,$$

$$p = 1 - \mathrm{LC}\omega^2, \qquad q = 1 - \mathrm{NK}\omega^2 \quad \text{and} \quad \mathrm{K} = \mathrm{K}_1 + \mathrm{K}_2.$$

It will be remembered that $\phi_1 = \delta_1/\pi$ and $\phi_2 = \delta_2/\pi$, where δ_1 is the log. dec. per complete period of the electrical circuit LRC, and δ_2 is the log. dec. of the resonator; q and p are small quantities compared with unity, and to a very close approximation may be considered equal to $2dn_0/n_0$ and $2dn_1/n_1$ respectively, where dn_0 is a small frequency difference from the frequency at which $q = 0$, and dn_1 has the same meaning with respect to p.

Now in the case of quartz ϕ_2 is very small indeed—as already anticipated it is of the order 30×10^{-6}—and as a result σ_1 will assume its minimum value, to a very close approximation, at that value of q at which the second term in the numerator vanishes. This is, when σ_1 is a minimum

$$q = -\frac{\mathrm{K}(\mathrm{C}+\mathrm{K}_2)}{\mathrm{CK}_t + \mathrm{K}_1\mathrm{K}_2}.$$

From an experimental standpoint, it is convenient to work in terms of frequency differences, and it is further very desirable to take as datum that frequency at which

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σ_1 has its minimum value, *i.e.*, the bottom of the crevasse. The equation can therefore be thrown into this form by substituting for q a quantity

$$q = -\frac{K(C + K_2)}{CK_t + K_1K_2} - \frac{2dn}{n_0}.$$

Equation (4) now becomes by a further slight simplification :—

$$\sigma_1^2 = \frac{\phi_1^2\left[\phi_2^2 + \frac{4dn^2}{n^2}\right]}{\left[\frac{2dn}{n_0}\frac{K_1K_2}{CK_t + K_1K_2} - p\frac{2dn}{n_0} - \frac{KCK_2^2}{(CK_t + K_1K_2)^2} - \phi_1\phi_2\right]^2 + \left[p\phi_2 - \frac{2dn}{n_0}\phi_1 - \phi_2\frac{K_1K_2}{CK_t + K_1K_2}\right]^2} \quad \ldots \quad (5)$$

This equation contains no approximations, and will accurately delineate the curve obtained for any condition of the electric circuit. We still retain the term p, which changes rapidly with change of frequency when L and C remain constant and are so adjusted that $LC\omega^2$ is approximately unity.

In testing the validity of the theorem by experiments, the form taken by equation (5) will depend upon the conditions under which the experiments have been carried out.

A number of cases may arise, as follows :—

(*a*) CADY'S Experiments.—At each adjustment of frequency to a known value, the capacity of the oscillatory circuit is adjusted so that, with respect to this adjustment, σ_1 is a maximum.

This case is met by substituting back into (5) the quantity $I - LC\omega^2$ for p, and then differentiating with respect to C considered as the variable. Having thus found an expression for the changes in C from some datum value, in terms of dn and the other constants, the quantity $C + dC$ must be substituted back into (5) in order to obtain an equation in which dn is the only variable. A very complicated expression results, which entails tedious computations, and does not lend itself well to the derivation of further equations for the values of $\sigma_{1\ \text{min.}}$ as a function of the rest of the constants of the circuits when these are varied. A fairly good approximation can be made, as shown later, by consideration of p as the only variable. This method of experimenting is, moreover, extremely laborious. A small rate of drift of the source will impair the accuracy of the results, owing to the long time taken to observe a complete curve of σ_1.

(*b*) Another case, corresponding somewhat with the above, is to adjust the condenser C so that at each frequency $LC\omega^2 = 1$, *i.e.*, $p = 0$.

This can be carried out fairly easily by disconnexion of the resonator from the oscillatory circuit, which is then tuned by adjustment of C to resonance at each frequency adjustment. This setting is best made by a two-reading method, since considerable

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accuracy is necessary. The observations of σ_1 are now somewhat more quickly made, and in general more accuracy is obtained than by method (*a*). The equation simplifies to

$$\sigma_1^2 = \frac{\phi_1^2\left[\phi_2^2 + \frac{4dn^2}{n^2}\right]}{\left[\frac{2dn\mathrm{K}\mathrm{K}_2}{n\mathrm{C}\mathrm{K}_t} - \frac{\mathrm{K}\mathrm{K}_2^2}{\mathrm{C}\mathrm{K}_t^2} - \phi_1\phi_2\right]^2 + \left[\frac{2dn\phi_1}{n} + \frac{\mathrm{K}_1\mathrm{K}_2\phi_2}{\mathrm{C}\mathrm{K}_t}\right]^2} \quad \ldots\ldots \quad (6)$$

In this equation $\mathrm{C}\mathrm{K}_t$ has been used instead of $\mathrm{C}\mathrm{K}_t + \mathrm{K}_1\mathrm{K}_2$, since in general the ratio $\mathrm{K}_1\mathrm{K}_2/\mathrm{C}\mathrm{K}_t$ is less than 0.005. It will be seen that the equation is simple, and that σ_1 approaches unity rapidly with increasing positive or negative values of dn.

Another still more convenient method of observation is as follows :—

(*c*) Having adjusted the frequency so that $\sigma_{1\,\mathrm{min.}}$ is obtained, the resonator is disconnected, and C is adjusted so that $\mathrm{LC}\omega^2 = 1$. The maximum value of σ_1 is adjusted to or called unity.

The complete curve of σ_1 against dn is then obtained without further adjustment of C. Under these conditions, $p = -2dn/n$ for the small frequency changes in question. Equation (5) then takes the form

$$\sigma_1^2 = \frac{\phi_1^2\left[\phi_2^2 + \frac{4dn^2}{n^2}\right]}{\left[\frac{4dn^2}{n^2} + \frac{2dn\mathrm{K}_1\mathrm{K}_2}{\mathrm{C}\mathrm{K}_t} - \frac{\mathrm{K}\mathrm{K}_2^2}{\mathrm{C}\mathrm{K}_t^2} - \phi_1\phi_2\right]^2 + \left[\frac{2dn(\phi_1+\phi_2)}{n} + \frac{\phi_2\mathrm{K}_1\mathrm{K}_2}{\mathrm{C}\mathrm{K}_t}\right]^2} \quad \ldots \quad (7)$$

In this equation $\mathrm{K}_1\mathrm{K}_2$ has again been neglected in comparison with $\mathrm{C}\mathrm{K}_t$.

(*d*) In the more general case, where C is left unvaried, we may set the oscillatory circuit at any frequency near that corresponding to (*a*).

It is convenient to write Δn for the frequency difference between that at which σ_1 is a minimum, *i.e.*, $q = -\mathrm{K}/\mathrm{K}_t$, and that at which $\mathrm{LC}\omega^2 = 1$, *i.e.*, $p = 0$.

We must then substitute in equation (5) $p = -2dn/n - 2\Delta n/n$, where Δn is a small frequency difference which does not vary when n is varied.

The equation for this case becomes

$$\sigma_1^2 = \frac{\phi_1^2\left[\phi_2^2 + \frac{4dn^2}{n^2}\right]}{\left[\frac{4dn^2}{n^2} + \frac{2dn}{n}\left(\frac{\mathrm{K}_1\mathrm{K}_2}{\mathrm{C}\mathrm{K}_t} + \frac{2\Delta n}{n}\right) - \frac{\mathrm{K}\mathrm{K}_2^2}{\mathrm{C}\mathrm{K}_t^2} - \phi_1\phi_2\right]^2 + \left[\frac{2dn(\phi_1+\phi_2)}{n} + \phi_2\left(\frac{\mathrm{K}_1\mathrm{K}_2}{\mathrm{C}\mathrm{K}_t} + \frac{2\Delta n}{n}\right)\right]^2} \quad \ldots\ldots \quad (7a)$$

This equation is useful when it is desired to carry out a series of experiments using various air gaps, as explained later. The frequency of response varies with variation of the air gap, as has been shown (Fig. 3). The use of (7*a*) enables us to leave C unchanged notwithstanding the change in response frequency. It is of course necessary to insert the value of Δn appropriate to each observed value of frequency at which $\sigma_{1\,\mathrm{min.}}$ occurs when the air gap is varied.

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In all the equations so far given for σ_1 it will be seen that the crevasse is not symmetrical, since the denominator has values depending upon the sign of dn. For the particular case in which C is so adjusted that $2\Delta n/n = -K_1K_2/CK_t$, the denominator contains only even-powered terms of dn, and hence does not depend upon its sign. For this case, therefore, the crevasse, when plotted to a scale of dn, is perfectly symmetrical.

This case is of interest chiefly in connexion with determinations of ϕ_2, since a very simple and accurate expression may be derived from it, as will appear immediately.

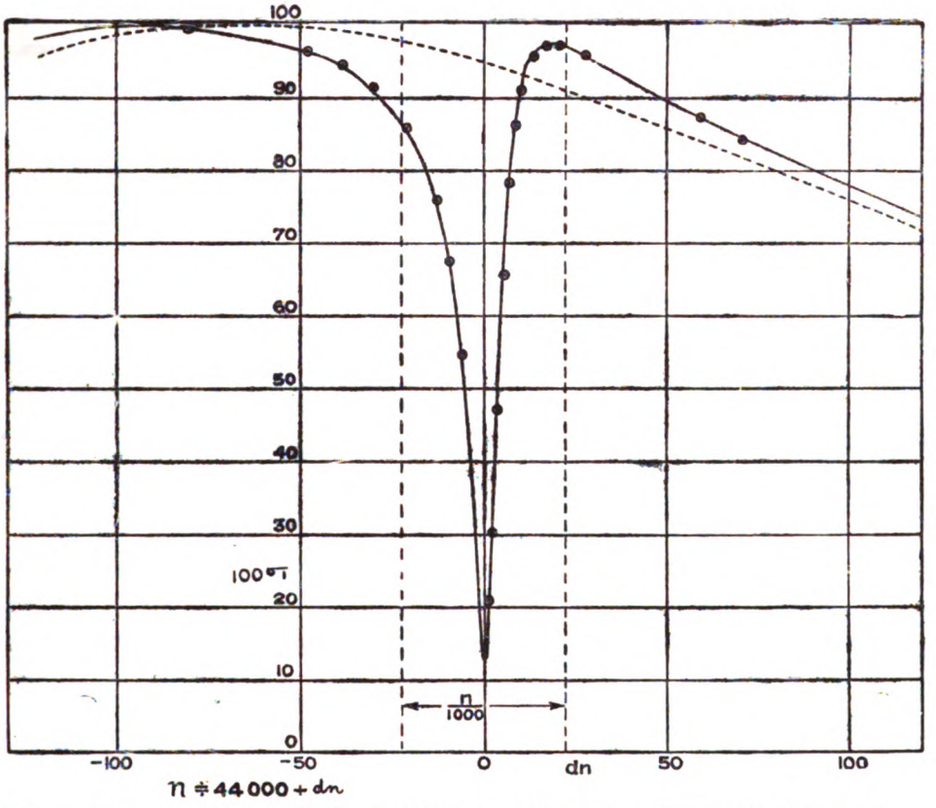

FIG. 6. Response Crevasse in Current Resonance Curve for a Longitudinal Resonator.

In order to be strictly accurate, we should consider ϕ_1 and ϕ_2 as variables when n is varied, since they contain ω. In practice, however, the range of frequency involved in the interesting part of the curves is so small that, so far as ω is concerned, ϕ_1 and ϕ_2 may be regarded as constant.

At this stage it is desirable to give some experimental and theoretical examples of curves in order to show the extraordinary accuracy of the agreement which may be obtained, and to make clear the effects on the crevasse of varying certain conditions. In Fig. 6 is an example of a curve of σ_1 for a resonator in the form of a bar approximately 6 cms. $\times$ 0.15 cm. $\times$ 0.6 cm., vibrating in its fundamental longitudinal mode at a frequency of approximately 44,000 cycles per second. This case has been chosen

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to bring out various points, and a suitable scale for the purpose has been adopted. The curve has been purposely made unsymmetrical by slightly detuning the electrical circuit, and so corresponds to case (*d*). The electrical circuit is in resonance at a frequency 64 cycles per second lower than the frequency at which σ_1 is a minimum. The flat dotted curve is the curve of current, I_0, in the oscillatory circuit that would have been obtained in the absence of the quartz. Owing to the open scale of frequency, the curve loses its more familiar shape, and what usually appears as a sharp peak is now a very rounded top. The whole belt of frequency covered on this curve is 0.5 per cent. wide. The constants of the circuit and the air gap have been chosen to give a very deep crevasse having a minimum value of 0.128.

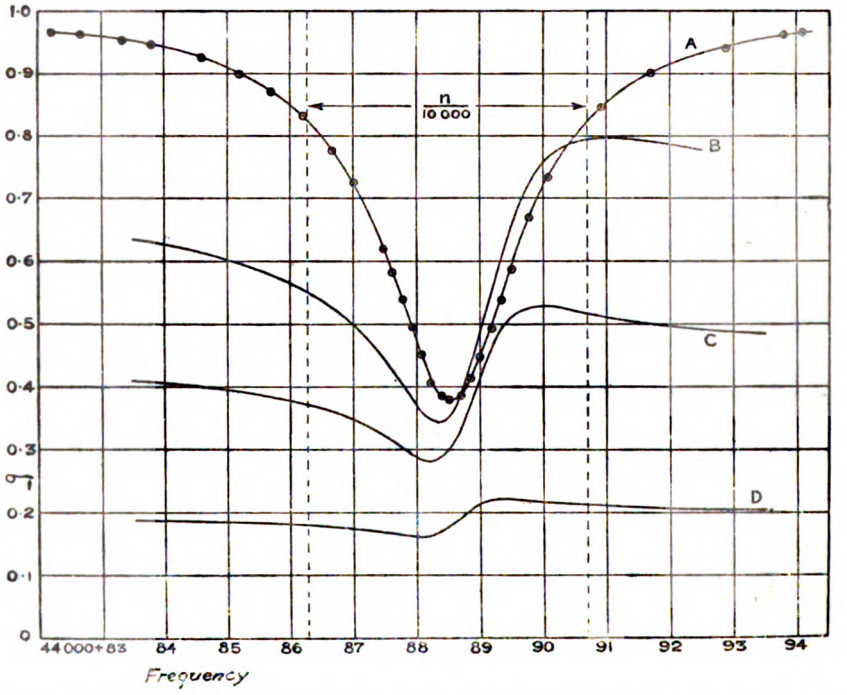

FIG. 7. Response Crevasse showing effect of Detuning the Resonant Circuit.

The curve has not been drawn through the points, but from an equation of the form (7*a*), and the observed points have been very carefully plotted—as dots in circles—afterwards. It will be seen that the agreement is very striking, and shows that within the accuracy possible experimentally the electrical behaviour of the resonator is exactly represented by the mesh of Fig. 2. The two parallel dotted lines show a belt of frequency of width one part in a thousand.

The next sheet of curves, Fig. 7, refers to the same resonator vibrating in the same mode as in Fig. 6. The air gap is slightly larger, the electrical circuit is such that the minimum is not so small as in the previous case, and the curves have been drawn on a very much more open scale of frequency in order to bring out the shape more clearly. The parallel dotted lines show a belt one part in ten thousand wide. In the case of curve

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A the circuit has been as nearly as possible experimentally tuned to exact resonance at the frequency of maximum response for a case where the quartz crystal is in position but prevented from vibrating. This corresponds to equation (7*a*), but the quantity $2\Delta n/n$ has been made equal to $-\mathrm{K_1K_2/CK_t}$, so that the curve should theoretically be quite symmetrical. Actually this condition is difficult to adjust quite exactly and so a small term equal to $\frac{\mathrm{K_1K_2}}{\mathrm{CK_t}} + \frac{2\Delta n}{n}$ has been included so as to fit the experimental curve at two suitable points. The departure from symmetry is inappreciable but can be seen by close inspection. The curve is again obtained from the equation and the observed points are again shown by dots in circles. The equation has been chosen to fit three of the points. The accuracy with which it fits the rest of them shows that the equation rightly accounts for the change in the curve consequent upon changing the electrical circuit.

The other curves are all theoretical and have been drawn to bring out the effects of detuning the oscillatory circuit. The curves in order B, C and D represent the currents that will be obtained when the circuit is tuned to frequencies 0.5, 1 and 2.5 per cent., respectively, lower than that of curve A. For these cases the crevasse is on the side of the ordinary resonance curve instead of being on the top as at A.

It will be seen that the position of $\sigma_{1\ \mathrm{min.}}$ is slightly altered as a result of the detuning. In the case of curve B, it is about five parts in a million lower than for curve A, whilst for curve D the crevasse is not well defined and occurs very low down on one side of the main resonance curve. The minimum here is displaced by about one part in a hundred thousand from that corresponding to the symmetrical curve.

It is thus seen that the frequency at which $\sigma_{1\ \mathrm{min.}}$ occurs is almost entirely independent of the precise adjustment of the electrical circuit to resonance. The setting of this circuit can, of course, be made much closer to resonance at the desired frequency than 0.5 per cent. Although it is not always easy to set it to one part in ten thousand it can always be set to one part in one thousand. For an uncertainty of this amount the uncertainty of the response frequency of the resonator will not be so great as three parts in a million.

The next curve (Fig. 8) represents a resonator of much higher frequency and of a different type. In this case the resonator is in the form of a small rectangular slab 1.564 cm. × 1.640 cm. × 0.615 cm. thick. This resonator is one which is suitable for the transverse mode of vibration. This particular piece was chosen with special care, is very homogeneous, and has a very small logarithmic decrement. It will oscillate readily when suitably connected up with a valve.

The curve corresponds to a nearly symmetrical case. The resonant frequency is approximately 472,000 cycles per second and the log. dec. is only 0.8×10^{-4}. The main part of the crevasse is comprised within the belt of 1 in 10,000 embraced by the dotted lines.

The remarkable agreement between the observed points (dots in circles) and the

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calculated curve shows that for a transverse resonator the theory also holds within the experimental limits. The equivalent electrical constants of a given piece of quartz are, of course, quite different according to whether it is vibrating in a longitudinal or in a transverse mode, but in general the log. dec. is about the same and the value of K_1 is not very different for the two modes of vibration.

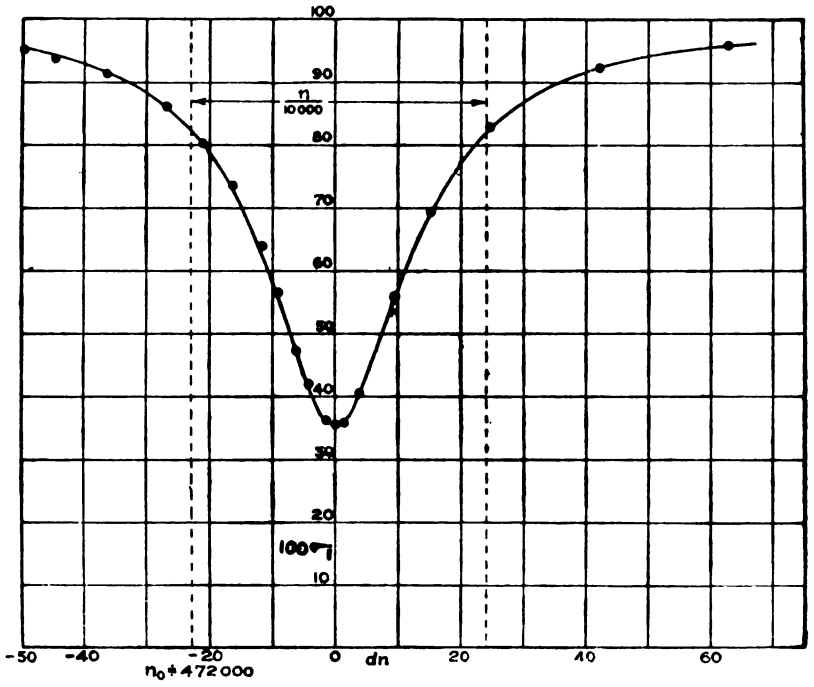

FIG. 8. Response Crevasse for a Transverse Resonator.

As might be expected, the damping coefficient for a short high-frequency resonator is not very different from that for a long low-frequency resonator.

As an extreme case, in Fig. 9 is given a curve for a resonator of no less a frequency than 15×10^6. This resonator is in the form of a small disc about 0·2 mm. thick and 6 mms. diameter. Its mode of vibration is transverse. Even at such a frequency the agreement between theory (drawn curve) and experiment (dots in circles) is satisfactory. In such a case the precise adjustment of the electrical circuit to resonance, at the bottom of the crevasse, is almost impossible owing to the smallness of the inductance and the capacity. The case shown is one in which the electrical circuit has a resonant frequency somewhat lower than the response frequency of the resonator. The experiments at such frequencies are difficult: special arrangements on the valve oscillator are necessary to enable smooth reversible changes of frequency of uncertainty less than one part in a hundred thousand to be made when the frequency is as high as 15×10^6 (20 metres wave-length).

For all the preceding curves a general form of equation (7*a*) may be used in cases

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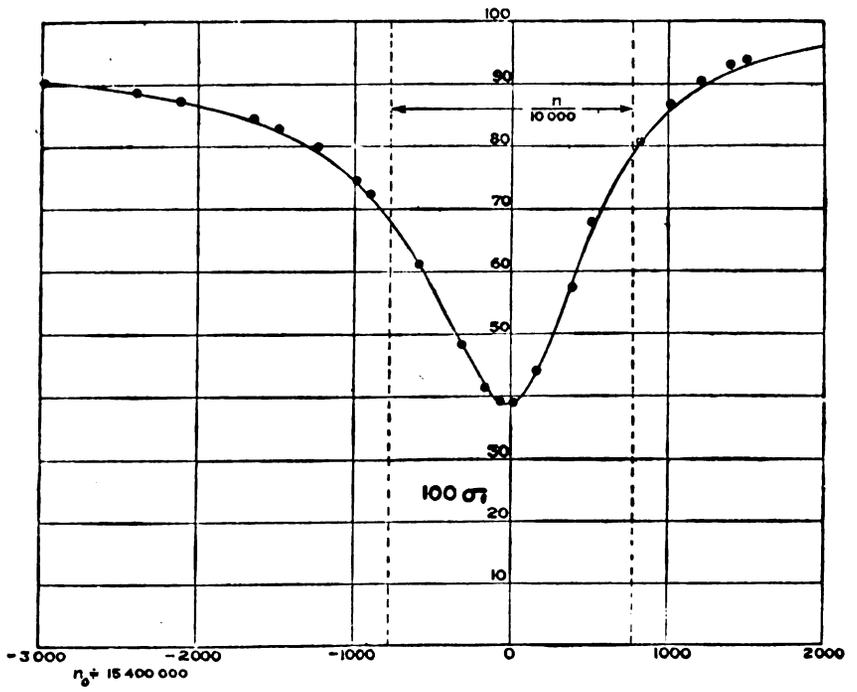

FIG. 9. Response Crevasse for a very High-Frequency Transverse Resonator.

where it is desired merely to plot a curve to see whether it fits the points observed. The equation may be written

$$\sigma_1^2 = \frac{A^2 + dn^2}{\left[\frac{2dn^2}{n\phi_1} + Bdn - \frac{A}{\sigma_{1\,\text{min.}}}\right]^2 + \left[dn\left(1 + \frac{2A}{n\phi_1}\right) + AB\right]^2}, \quad \ldots \quad (8)$$

in which $2/n\phi_1$ belongs only to the electrical circuit, whilst A belongs only to the quartz and is equal to $n\phi_2/2$. The quantity B depends upon the frequency difference between the electrical circuit and the quartz—whilst $\sigma_{1\,\text{min.}}$ is determined by the piezo-electric reaction of the quartz upon the electrical circuit.

We have now to consider how we can use equations (7) and (7*a*) and derivations from them in order to evaluate the quantities K, S and K_1.

Determination of $\phi_2 = SK\omega$.—The quantities which we may vary are (*a*) ω; (*b*) $\phi_1 = RC\omega$ (by addition of resistance in the oscillatory circuit); (*c*) C; and (*d*) K_2 (by variation of the air gaps). We can observe curves of σ_1 as a function of ω in the way already shown in Figs. 6 to 9 for various known values of ϕ_1, C and K_2.

By observation of the width of the crevasse in terms of the frequency difference at various chosen values of σ_1, as read off on the plotted curve, we can first determine ϕ_2 as follows:—

Consider the particular case of (7) when $dn = 0$. The equation becomes

$$\sigma_1^2{}_{\text{min.}} = \frac{\phi_1^2\phi_2^2}{\left[\frac{KK_2^2}{CK_t^2} + \phi_1\phi_2\right]^2 + \left[\frac{K_1K_2\phi_2}{CK_t}\right]^2}. \quad \ldots \quad (9)$$

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The second term in the denominator has very little effect on $\sigma_{1\,\text{min.}}$ To a first approximation, therefore, we may neglect it. We can then substitute back into equation (7) the term $\phi_1\phi_2/\sigma_{1\,\text{min.}}$ in place of $KK_2^2/CK_t^2 + \phi_1\phi_2$. By confining attention to that part of the curve corresponding to values of σ_1 not greater than 0·8, terms containing dn^3 and dn^4 may be entirely neglected. On solving for ϕ_2 it will be found that

$$\phi_2 = \frac{2dn\sigma_{1\,\text{min.}}}{n}\sqrt{\frac{1-\sigma_1^2}{\sigma_1^2-\sigma_{1\,\text{min.}}^2}}\left\{1+\tfrac{1}{2}\frac{\sigma_1^2}{1-\sigma_1^2}\cdot\frac{K_2^2}{\phi_1^2K_t^2}\left(\frac{2K}{C}-\frac{K_1^2}{C^2}+\frac{nKK_1K_2}{dnC^2K_t}\right)\right\}. \quad (10)$$

The terms in the round brackets are correction terms.
Now this equation may be written in the form

$$\phi_2 = f[\sigma_1][dn] - f'[\sigma_1], \quad \ldots\ldots\ldots\ldots \quad (11)$$

in which $f'[\sigma_1]$ is a small quantity compared with $f[\sigma_1][dn]$. But ϕ_2 is also positive when dn is negative, and has a value, say, dn'. Measuring dn' to the left, therefore, we have

$$\phi_2 = f[\sigma_1][dn'] + f'[\sigma_1]. \quad \ldots\ldots\ldots\ldots \quad (12)$$

If, therefore, we measure the frequency difference $\delta n = dn + dn'$ right across the crevasse as shown in Fig. 10, at the level **AB** corresponding to any chosen value of σ_1, we shall have

$$2\phi_2 = f[\sigma_1][\delta n]. \quad \ldots\ldots\ldots\ldots \quad (13)$$

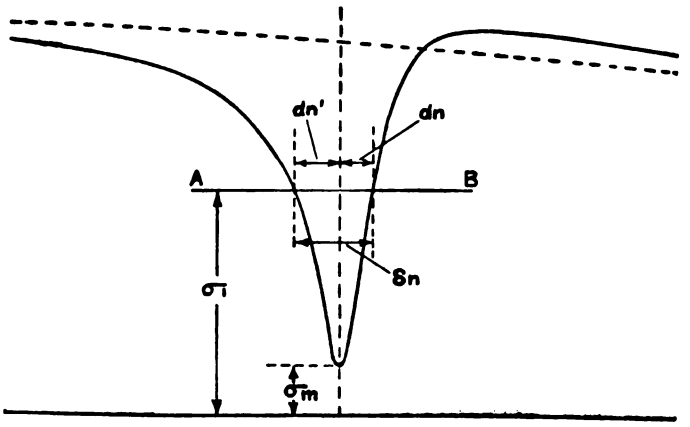

FIG. 10. Diagrammatic Representation of a Response Crevasse.

The equation for ϕ_2 then becomes

$$\phi_2 = \frac{\delta n\sigma_m}{n}\sqrt{\frac{1-\sigma_1^2}{\sigma_1^2-\sigma_{1\,\text{min.}}^2}}\left\{1+\tfrac{1}{2}\frac{\sigma_1^2}{1-\sigma_1^2}\cdot\frac{K_2^2}{\phi_1^2K_t^2}\left(\frac{2K}{C}-\frac{K_1^2}{C^2}\right)\right\}. \quad \ldots \quad (14)$$

In this equation K and K_1 are unknown, but as they only occur in a correction term of the order of 1 or 2 per cent., we may assume a value for K_1 based on a value of 4 for the dielectric constant of quartz. The quantity K (as yet unknown) can be eliminated

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by substitution of its equivalent in the approximate form of equation (8). This gives

$$\phi_2 = \frac{\delta n \sigma_{1\,\text{min.}}}{n} \sqrt{\frac{1-\sigma_1^2}{\sigma_1^2 - \sigma_{1\,\text{min.}}^2}} \left\{1 + \frac{\sigma_1^2}{\phi_1(1-\sigma_1^2)} \left(\frac{1-\sigma_{1\,\text{min.}}}{\sigma_{1\,\text{min.}}} \phi_1\phi_2 - \frac{K_1^2 K_2^2}{2C^2 K_t^2}\right)\right\}. \quad (15)$$

This equation gives a very accurate measure of ϕ_2 from observations of δn read off at various values of σ_1 from the curve.

For most purposes, however, the correction term may be neglected and the very simple expression (16) used in the determination of ϕ_2.

$$\phi_2 = \frac{\delta n \sigma_{1\,\text{min.}}}{n} \sqrt{\frac{1-\sigma_1^2}{\sigma_1^2 - \sigma_{1\,\text{min.}}^2}}. \quad \ldots\ldots\ldots \quad (16)$$

In thus determining ϕ_2 from a curve, the best way is to read off a series of values of δn corresponding to a number of values of σ_1. By plotting $\sqrt{(\sigma_1^2 - \sigma_{1\,\text{min.}}^2)/(1-\sigma_1^2)}$ ordinate to δn as abscissa, a straight line should result, of slope m such that $\phi_2 \doteqdot \frac{m\sigma_{1\,\text{min.}}}{n}$.

As an example of the determination of ϕ_2 in this manner, the curve in Fig. 11 has been plotted for the case of the curve of which the experimental points are given in

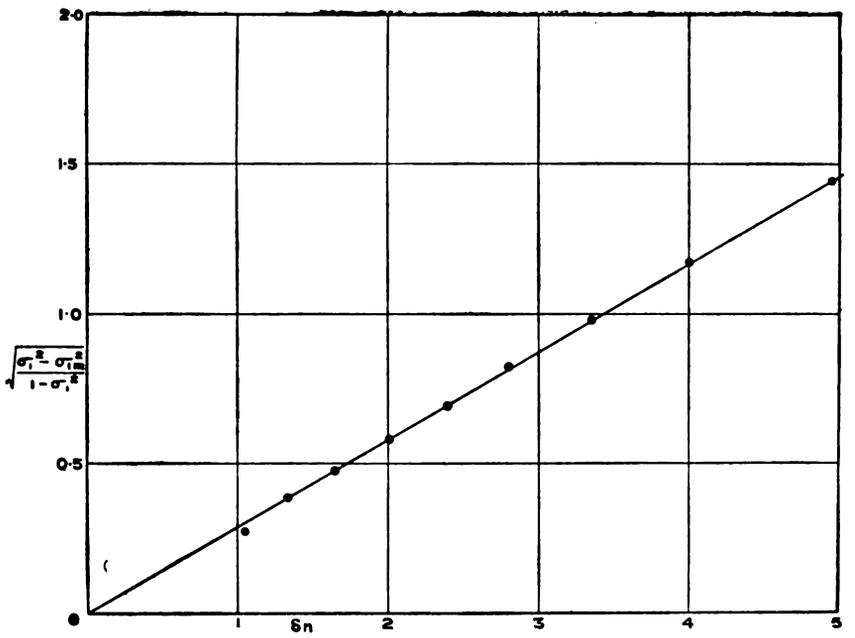

FIG. 11. Linear Relationship deduced from Crevasse whereby the log. dec. of a Resonator may be obtained.

Fig. 7. It will be seen that the points lie on a very good line. The value of ϕ_2 obtained from the line is 29.7×10^{-6}, whence the logarithmic decrement of the quartz $= \pi \times 29.7 \times 10^{-6} = 0.93 \times 10^{-4}$.

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The measurement of ϕ_2 constitutes the most important single measurement that can be made on a quartz resonator.

Having determined ϕ_2 we are in a position to reproduce mathematically any curve for a resonator and a circuit in which ϕ_1 is known, without further analysis. This is seen from the generalised equation (8) which, except for the quanity B, does not entail any knowledge of the resonator other than its damping. The term B only enters into the equation in so far as it effects the symmetry of the curve. It has been seen already by equation (7*a*) that B is very largely dependent upon the small difference between the resonant frequencies of the resonator and the electrical circuits. The deduction of B from the asymmetry of the curve cannot, therefore, safely be used for the purpose of further analysing the resonator circuit.

We are therefore left with $\sigma_{1\ \mathrm{min.}}$ only, as a particular property of the observed curve, as a means whereby a further analysis may be made.

Returning to equation (9) in its simplified form, we have, to a close approximation,

$$\sigma_{1\ \mathrm{min.}} \doteqdot \frac{\phi_1\phi_2}{\frac{KK_2^2}{CK_t^2} + \phi_1\phi_2}. \qquad \ldots\ldots\ldots\ldots (17)$$

By substitution of $SK\omega$ for ϕ_2 we can eliminate the unknown quantity K and obtain the equation

$$\sqrt{\frac{\sigma_{1\ \mathrm{min.}}}{1-\sigma_{1\ \mathrm{min.}}}} = \sqrt{\phi_1 SC\omega}\left(1+\frac{K_1}{K_2}\right) \ldots\ldots\ldots (18)$$

The quantities we can vary are :—

K_2	by adjustment of the air gaps to known values.
$\phi_1 = RC\omega$	by addition of resistance in the oscillatory circuit.
C	by choice of capacity and inductance in the oscillatory circuit.

Since C and ϕ_1 occur in exactly the same way in the equation, nothing is gained by varying both of them. In practice it is much easier to vary ϕ_1, since this does not necessitate retuning and does not react on C.

If a series of curves of σ_1 are taken for various values of K_2, and also with various added resistances in the oscillatory circuit, we can observe the following quantities on these curves :—

(*a*) The various values of ϕ_2.
(*b*) The various values of $\sigma_{1\ \mathrm{min.}}$.
(*c*) The frequencies at which the various values of $\sigma_{1\ \mathrm{min.}}$ occur.

Observations (*a*) tell us whether the logarithmic decrement of the resonator is independent of the conditions under which it resonates.

Observations (*c*) will be dealt with later, as they give peculiar results which will repay further investigation.

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From observations (*b*) we can find S and K_1 by means of equation (18).

Since ϕ_2 is also known, we can determine K. Finally, N is determined from the relation $NK\omega^2 = 1$.

The procedure is to obtain a family of curves each corresponding to a known air gap, and hence to a known value of K_2. The values of $\sigma_{1\ min.}$ are then read off. In carrying out these observations it is, of course, not really necessary to obtain the whole of the crevasse, but by way of illustration the family of curves shown in Fig. 12 has been taken, which are fairly complete, and enable a number of interesting facts to be seen. In

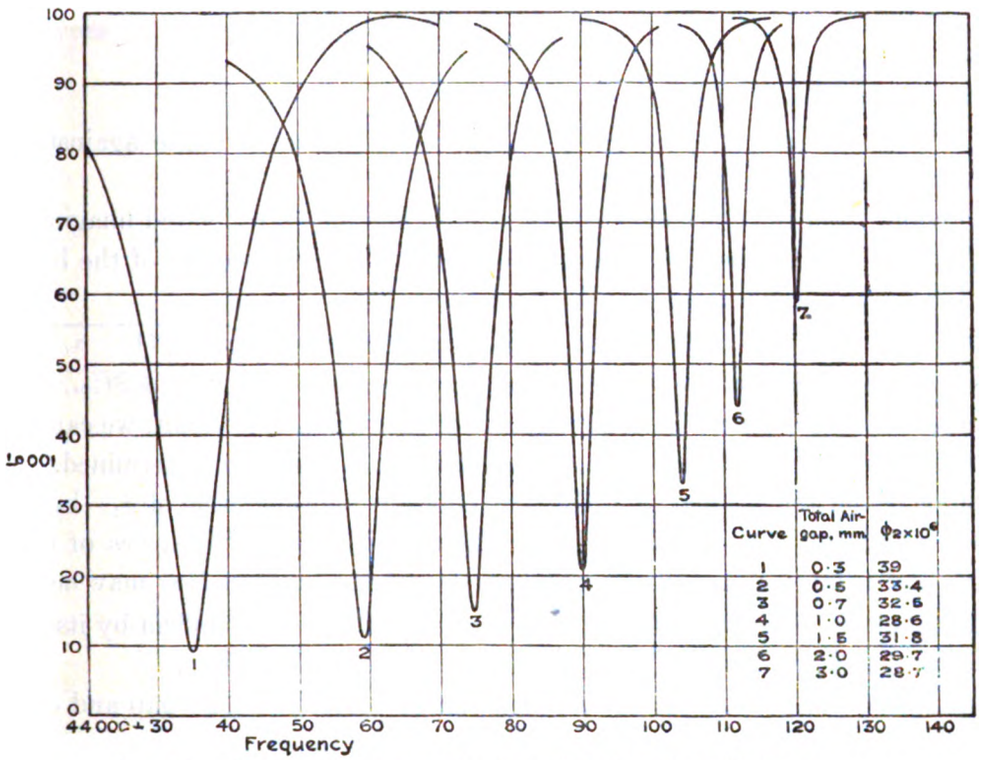

FIG. 12. Family of Crevasses showing effect of various Air Gaps.

order to avoid confusion, the observed points have been omitted. The curves are experimental, and not theoretical.

Each crevasse corresponds to a particular air gap, and this is the only quantity which has been varied, except for the small change in C necessary to establish resonance in the electrical circuit at each frequency of response. It will be seen that the frequency of response increases regularly as the air gap is increased, as indicated by the theory represented in Fig. 4. The values of $\sigma_{1\ min.}$ also lie on a smooth curve, representing a reduced effect upon the oscillatory circuit as the air gap is increased. It has been necessary to draw these curves on a rather compressed scale of frequency owing to the relatively large changes in response frequency with change in air gap. From observations made on each curve when drawn on a more open scale the value of ϕ_2 was deduced.

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ϕ_2 was found to vary somewhat irregularly, as will be seen by reference to the table on Fig. 12, giving this quantity for the various crevasses. There seems to be a slight increase in ϕ_2 as the air gap is diminished, but this is not regular. It was afterwards found to be due to variations in either the mounting of the resonator or the humidity of the air. In a latter series only a small portion of the crevasse near $\sigma_{1\ \text{min.}}$ was taken, so that all the necessary observations might be made on one occasion. The mounted resonator was placed in a bell jar and kept dry during this series of observations.

Returning now to equation (18), we can write this in the form

$$\sqrt{\frac{\sigma_{1\ \text{min.}}}{1-\sigma_{1\ \text{min.}}}} = A_1 + \frac{A_1 K_1}{K_2}, \quad \ldots\ldots\ldots\ldots (18a)$$

where $A_1 = \sqrt{\phi_1 SC\omega} = \sqrt{RSC^2\omega^2}$.

It will be seen that on plotting $\sqrt{\sigma_{1\ \text{min.}}/(1-\sigma_{1\ \text{min.}})}$ as ordinate against $1/K_2$ as abscissa we should obtain a straight line.

It is obvious from the equation that when $1/K_2$ has the extrapolated imaginary value of $-1/K_1$ the left-hand side of (18) becomes zero. The intersection of the line on the axis of $1/K_2$ gives us therefore $-1/K_1$, and hence K_1.

Again, when $K_2 = \infty$, and hence $1/K_2 = 0$, we have $\sqrt{\sigma_{1\ \text{min.}}/(1-\sigma_{1\ \text{min.}})} = A_1$. The intercept of the line on the ordinate axis therefore gives us $A_1 = \sqrt{\phi_1 SC\omega}$. Assuming that we have measured the effective resistance of the oscillatory circuit, we can calculate ϕ_1. Of the other terms, C and ω are known, and hence S may be determined.

The slope of the line is equal to $A_1 K_1$. By choosing another value of ϕ_1—by inserting a known resistance in the oscillatory circuit—another family of curves of σ_1 can be obtained, and from it a second straight line satisfying equation (18) may be deduced. This line should have a slope proportional to $\sqrt{\phi_1}$, and should confirm by its intercept on the axis of $1/K_2$ the value of $1/K_1$ obtained by the previous line.

A confirmation of both S and K_1 with respect to variation of air gap and of ϕ_1 may therefore be obtained by observing several series of crevasses, each crevasse in a series corresponding to a known air gap and each series corresponding to a chosen value of ϕ_1. The deduced lines of equation (18*a*) should all meet at the point corresponding to $1/K_1$, and their intercepts on the ordinate axis should be proportional to $\sqrt{\phi_1}$ if S has remained constant. An overall average value of S may therefore be obtained by plotting the squares of the intercepts against ϕ_1, when a line will be obtained whose slope is equal to $SC^2\omega^2$.

An alternative method of dealing with equation (13) when deducing S and K_1 is to consider it in the form

$$\frac{\sigma_{1\ \text{min.}}}{1-\sigma_{1\ \text{min.}}} = RSC^2\omega^2 \frac{K_2^2}{K_t^2}. \quad \ldots\ldots\ldots\ldots (19)$$

By plotting $\sigma_{1\ \text{min.}}/(1-\sigma_{1\ \text{min.}})$ against R, or against "added R," in the oscillatory circuit we can get a series of straight lines each corresponding to a constant value of K_2. These

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lines should, in the "added R" case, intersect on the axis of "added R" at a point $-R_0$, where R_0 is the effective resistance of the oscillatory circuit when "added R" $= 0$.

The slopes of the lines are equal to

$$SK_2^2C^2\omega^2/K_t^2 = F^2SK_2^2/K_t^2 = m_1, \quad \ldots\ldots\ldots\ldots (20)$$

where

$$F = C^2\omega^2.$$

If, therefore, we plot $\sqrt{m_1}$ against $1/K_2$ as before, we shall obtain the intercept giving K_1 and that giving S. The latter does not now involve ϕ_1, but only C and ω, and the method averages all the observations.

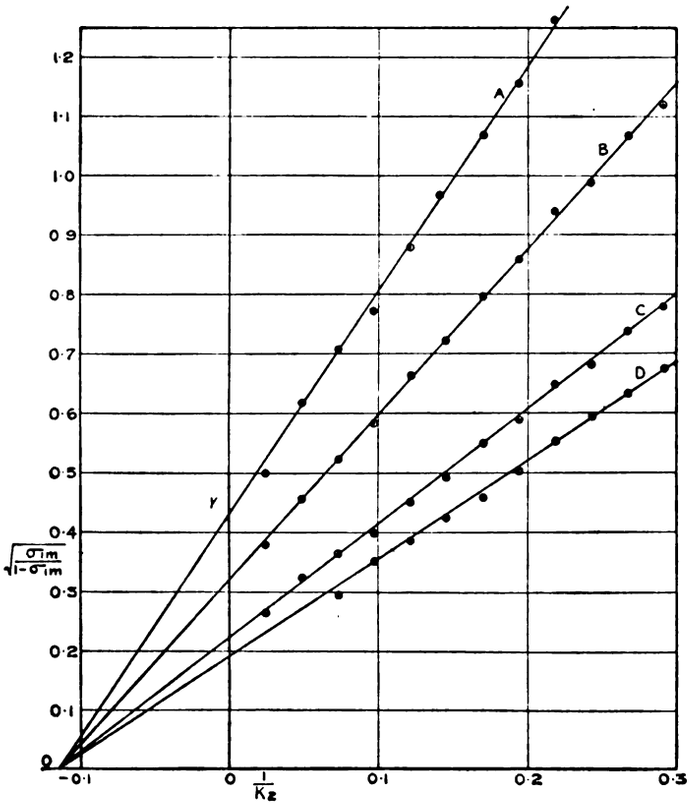

FIG. 13. Linear Relationship obtained from Fig. 12, whereby the Resonator Network may be analysed.

The foregoing methods of analysis have been applied to a number of resonators. The resonator for which the family of curves of σ_1 is shown in Fig. 12 has been completely analysed in this manner.

Three sets of curves similar to those of Fig. 12 were taken for three different total resistances in the oscillatory circuit. The values of $\sqrt{\sigma_{1\,\mathrm{min.}}/(1-\sigma_{1\,\mathrm{min.}})}$ were evaluated in each set at the corresponding values of K_2 and the resulting lines plotted in accordance with equation (18).

These lines are given in Fig. 13. The lines are the best straight lines through the

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points observed—shown as dots in circles. A considerable number of points were taken in each case. It will be seen that the points lie on the lines to a good approximation, and that the lines converge with very good agreement to a point of intersection on the abscissa axis. The data respecting these lines are given in the following table :—

TABLE I.—Resonator.

Bar 6.22 cms. long by 0.15 cm. thick (direction of electric axis) by 0.75 cm. deep (optic axis). Vibrating in its fundamental longitudinal mode.

Curve.	Total resistance in oscillatory circuit.	$\phi_1 \times 10^2$.	Intercept OY.	S ohms.	K_1 $\mu\mu$F.
A	34·2	6·2	0·43	1,650	8·7
B	19·3	3·48	0·320	1,635	8·7
C	9·4	1·70	0·224	1,640	8·7
D	4·70	0·85	0·19	2,360	8·7

It will be seen that in the case of lines A, B, C, the value of S is constant to within less than 1 per cent. Line D was taken afterwards in order to bring out a special point.

It was soon found in these experiments that the damping was variable from day to day. This was found to be due to two causes : (*a*) the mode of support, and (*b*) the state of the surface of the resonator with respect to cleanness and dryness. The experiments resulting in lines A, B and C were very carefully carried out. The resonator, after careful cleaning, was placed in an air-tight enclosure, and dry air was pumped through it after each adjustment had been made to the resonator. The experiments D were carried out without the enclosure and on a damp day. The increase in S is very marked, and is probably largely due to an increase in the electrical losses on the surface of the quartz. It will be observed that the line passes through the same value of K_1, showing that this constant does not appear to be affected by the moisture in the air.

In order that these particular experiments might be carried through in a single series, attention was concentrated upon observations of $\sigma_{1\,\text{min.}}$, and not on the delineation of the whole of the σ_1 curve. Consequently, values of ϕ_2 for these actual conditions were not obtained. If, however, the evidence of the tests made in moist air is taken for the purpose of the assumption that K_1 is not dependent on ϕ_2, we can make use of any of the curves of σ_1 taken at any time, notwithstanding ϕ_2, and hence S may be different for different curves. We can do this, of course, because K_1 may be considered as known and constant in equation (8). For the purpose of obtaining K the equation takes the convenient form

$$K = \phi_1\phi_2 C\left(1 + \frac{K_1}{K_2}\right)^2\left(\frac{1 - \sigma_{1\,\text{min.}}}{\sigma_{1\,\text{min.}}}\right). \quad \ldots\ldots\ldots \quad (21)$$

Using this equation on about 15 different curves of σ_1 for cases in which ϕ_1, ϕ_2 and K_2 have each been varied over wide ranges, results have been obtained as shown in Table II.

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TABLE II.—Longitudinal Resonator.

Dimensions, 6.22 cms. long, 0.15 cm. thick, 0.75 cm. deep. Fundamental mode of vibration at 44,000 cycles per second.

Total air gap, mms.	K_2 $\mu\mu F$.	$\phi_1 \times 10^3$.	$\phi_2 \times 10^6$.	$\sigma_{1\,min.}$	K $\mu\mu F$.
0·3	$13·7_3$	8·7	$41·0_8$	$0·09_0$	$0·063_0$
0·3	$13·7_3$	17·0	$41·0_4$	$0·17_4$	$0·058_1$
0·3	$13·7_3$	8·5	$39·0_0$	$0·09_1$	$0·057_7$
0·5	$8·2_4$	8·5	$33·0_4$	$0·11_0$	$0·063_2$
0·5	$8·2_4$	20·6	$31·0_3$	$0·19_8$	$0·059_5$
0·7	5·8	8·5	$32·0_5$	$0·15_0$	$0·062_8$
0·7	5·8	8·7	$40·0_3$	$0·18_6$	$0·061_5$
1·0	$4·1_2$	8·5	$28·0_6$	$0·20_9$	$0·058_2$
1·0	$4·1_2$	8·7	$40·0_2$	$0·26_9$	$0·060_0$
1·0	$4·1_2$	20·6	$27·0_2$	$0·37_0$	$0·060_5$
1·5	2·75	8·5	31·8	$0·33_0$	$0·061_8$
1·5	2·75	8·7	40·6	$0·40_7$	$0·058_3$
2·0	$2·0_6$	8·5	29·7	$0·44_0$	$0·057_3$
3·0	1·37	8·5	28·7	$0·58_8$	$0·060_2$
3·0	1·37	8·7	44·0	$0·68_3$	$0·062_4$
				Average value	0·0603

Mean variation from mean = 2·7 per cent.

The foregoing analysis applies to a longitudinal resonator. It is interesting to see whether the same general agreement with theory may be obtained in the case of a piece of quartz vibrating transversely. A similar set of experiments was therefore carried out on the small rectangular piece for which a specimen curve of σ_1 has been given in Fig. 8. A suitable gap was constructed from an ordinary 25 mm. micrometer gauge, and a careful series of curves of σ_1 for various air gaps and for various values of ϕ_1 were taken as before.

As will appear later on, remarkable results are obtained with transversely vibrating resonators under certain adjustments of the air gap. The series of curves was specially chosen in the light of this experience, which explains the apparently irregular air gaps chosen. The results of these experiments are shown in Fig. 14. By way of variation, the lines have been plotted in accordance with equation (19). Thus, taking square roots of each side, $\sqrt{\sigma_{1\,min.}/(1-\sigma_{1\,min.})}$ has been plotted as ordinate to an abscissa which is a fraction of the total resistance in the circuit. Actually the quantity plotted is $\sqrt{\phi_1}$. Each line A, B, C, etc., now corresponds to a definite air gap. The points on the line are obtained by calculation from the observed values of $\sigma_{1\,min.}$ obtained for various values of added resistance to the oscillatory circuit. By the help of the measurement of the effective resistance of the circuit when no resistance is added the value of ϕ_1 was calculated in each case. The lines of Fig. 14 correspond in order to air gaps of 0.2, 0.4, 0.9, 1.2 and 1.8 mm., for which the capacities K_2 are 11.35, 5.67, 2.52, 1.88 and 1.26 $\mu\mu F$ respectively.

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It will be seen that good proportionality exists between ϕ_1 and $\sigma_{1\text{min.}}/(1 - \sigma_{1\text{min.}})$. The slopes of these lines give us a quantity m_2, such that

$$m_2 = \sqrt{\sigma_{1\text{min.}}/\phi_1 \, (1 - \sigma_{1\text{min.}})} = (1 + K_1/K_2) \sqrt{\phi_2 C/K}.$$

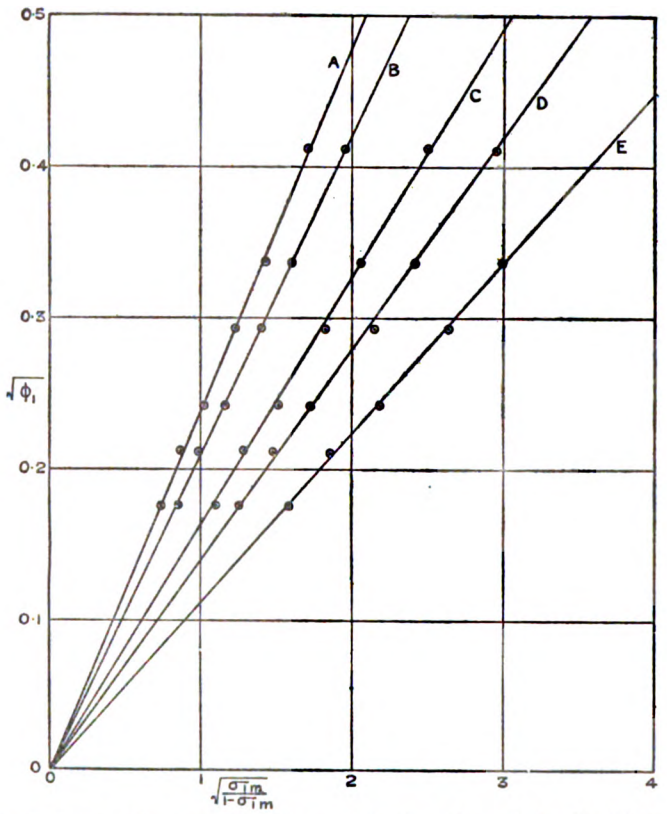

FIG. 14. Linear Relationship as a Function of the log. dec. of the Oscillatory Circuit.

It will be seen that on plotting m_2 against $1/K_2$ we get a mean line similar to those of Fig. 13.

The necessary measurement of ϕ_2 for each air gap was carried out for the condition "added R" = 0. The values obtained varied considerably, and appeared to diminish with increasing air gap. In the light of much more careful experiments made to investigate the effects of air gap on damping, remarkable effects were obtained, and for this series of experiments it is probable that an average of the values of ϕ_2 represents the truth more nearly than the assumption that the values obtained are each correct. In order of increasing air gap these values were $\phi_2 = 27.1$, 26.7, 25.6, 25.0 and 24.0×10^{-6} in each case.

The line of Fig. 15 represents the values of $m_2/\sqrt{\phi_2}$ plotted against $1/K_2$. The intercept on the axis $m_2/\sqrt{\phi_2}$ gives $\sqrt{C/K}$, from which K is determined. The intercept on the axis of $1/K_2$ gives $-1/K_1$. From the line the actual values are $K = 0.0041$ $\mu\mu$F and $K_1 = 1.93$ $\mu\mu$F. If the observed values of ϕ_2 are used the dotted line is obtained,

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which appears to be slightly steeper and gives for K and K_1 the values 0.0041 $\mu\mu$F and 2.00 $\mu\mu$F respectively.

Effect of Air Gap on Frequency.—We have seen in the simple circle diagram of the resonator that according to the theory the response frequency is a different quantity from what has been defined as the true resonant frequency, *i.e.*, that frequency at which $NK\omega_0^2 = 1$. We have seen that the response frequency at which σ_1 has its minimum value occurs, to a very close approximation, when the effective capacity K_0 of the resonator mesh is equal to $-K_2$. As already noted in equation (4), this means that $\sigma_{1\,\text{min.}}$ occurs when $q = -K/(K_1 + K_2)$.

We have, therefore, $1 - NK\omega^2 \doteqdot 2(n_0 - n)/n_0$ for any value of n near n_0.

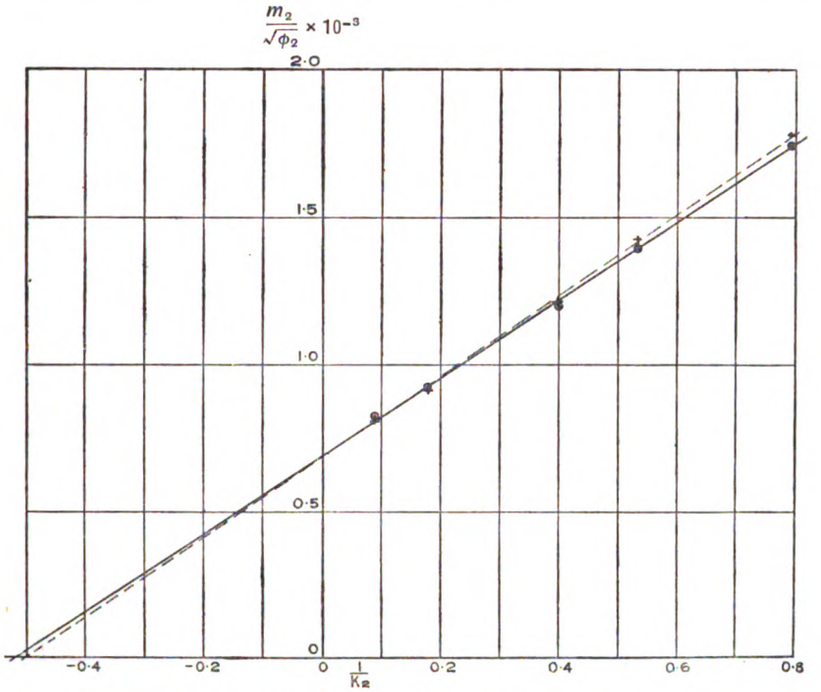

FIG. 15. Analysis of Resonator Network from Observations on the lines of Fig. 14.

This gives the relation

$$n_{\sigma_1\text{min.}} = n_0 + n_0 K/2(K_1 + K_2), \quad \ldots\ldots\ldots\ldots (22)$$

where $n_{\sigma_1\text{min.}}$ is the frequency at which σ_1 has its minimum value. By observation of $n_{\sigma_1\text{min.}}$ at various values of K_2 we ought, therefore, to obtain a check on the value of K or to discover whether K varies slightly with air gap. Using the determined value of K_1 we can plot $n_{\sigma_1\text{min.}}$ against $1/(K_1 + K_2)$. A straight line should result having a slope $n_0 K/2$, and an intercept n_0 on the axis of n.

A great number of measurements of the response frequency have been made on various resonators as a function of air gap, and a number of very interesting results have been obtained. A typical case is given in the curve of Fig. 16 for the longitudinal resonator of which the analysis has already been given. The values of response frequency are plotted against the corresponding values of $1/(K_1 + K_2)$. For this purpose the

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value 8.7 $\mu\mu$F already determined has been used for K_1. The vertical dotted line represents the case for an infinite air gap, for which $K_2 = 0$, and hence gives a limiting value beyond which n cannot rise. A thin line of slope to fit the equation $2dn/n = K/K_t$ has been drawn to pass through the point A on the experimental curve. Values of 0.0603 $\mu\mu$F and 8.7 $\mu\mu$F respectively have been used for K and K_1 in calculating this line.

It is seen that the frequency change does not accurately obey the simple theory, but that it tends to do so for large air gaps, where it is practically coincident with the line.

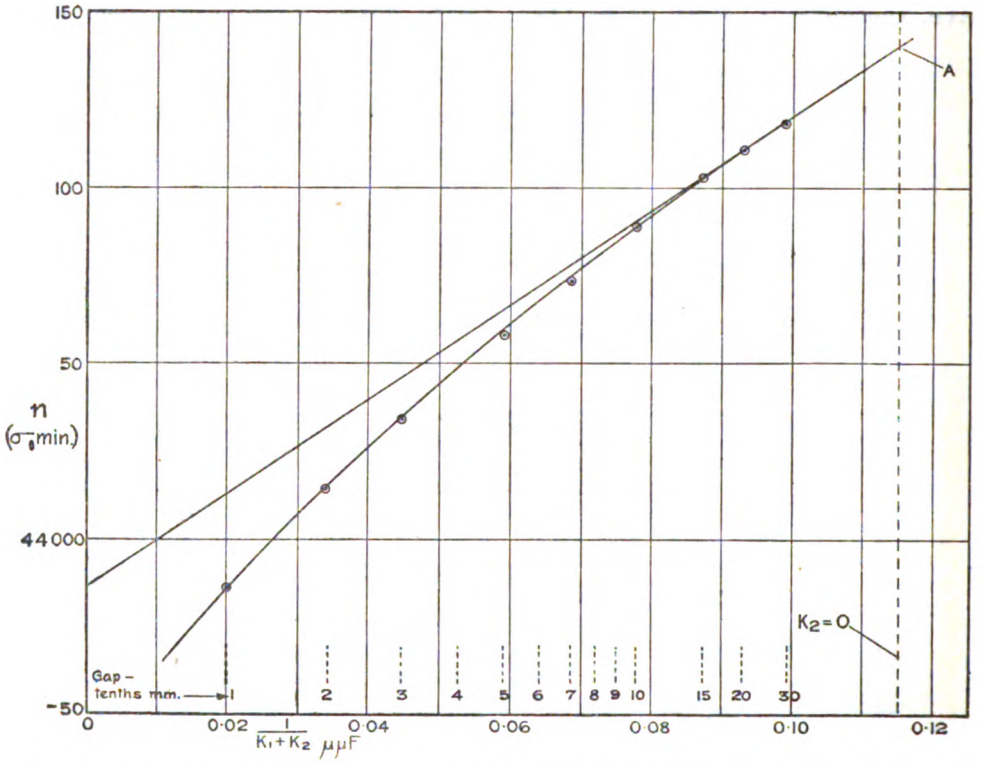

FIG. 16. Variation of Response Frequency with Variation of Air Gap for a Longitudinal Resonator.

CADY has already indicated that a change of the resonant frequency of a bar is to be expected when the air gap is varied, and suggests that this is due to the leakage electric field resulting from the vibration of the bar. The suggestion is that the vibrating bar is analogous to a bar magnet, and that the leakage field reacts upon the modulus of elasticity, thus altering the true resonant frequency. It is probable that this reasoning is true in part, but it is only necessary to assume it in respect of the departure of the experimental values of n from those corresponding to the straight line of slope $Kn_0/2$, and must not be considered as the main cause of the frequency change.

It may be quite definitely stated that, except for very small air gaps (less than 0.1 mm.), the main change in frequency consequent upon an alteration in the air gap is due to the fact that the resonator behaves as an inductance of such value as to give the equivalent

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electrical resonance corresponding to the capacity K_2 of the air gap. This is a very interesting fact, and requires further investigation, using electrodes of smaller length than that of the bar, with a view to the experimental determination of the real change in the true resonant frequency as a function of air gap.

It will be seen that the possible change in the frequency of a resonator due to adventitious variation of the air gap is by no means negligible. On the other hand, such a possibility of varying the air gap affords a convenient means of finely adjusting the frequency to a desired value when once the resonator has been ground to within one or two parts in a thousand in frequency. There is a range of about three parts in a thousand in the case of a longitudinal resonator for reasonable limits of the air gap.

To a sufficient approximation the change in frequency produced by an increase in air gap from a total of t_1 to t_2 will be given by the expression

$$\frac{\Delta n}{n} = \frac{kt}{2\mu}\left[\frac{t_2 - t_1}{(kt_1 + t)\,(kt_2 + t)}\right]$$

where

t is the thickness of the quartz,

k is its dielectric constant (approx. 4.3),

and

μ is the piezo-electric ratio K_1/K.

For an average good specimen the quantity μ is of the order 150 for the transverse effect which produces the longitudinal force. For a good plate vibrating transversely the value of μ is of the order 400 to 500. In the case of a resonator vibrating transversely remarkable effects are obtained when the air gap is varied.

Take, for example, the transverse resonator for which the analysis has been given. For a narrow range of air gap extending from a very small total gap up to one of a few tenths of a millimetre, the change in frequency is smooth and of the same general character as that shown by the longitudinal resonator. The increase in $\sigma_{1\,\text{min.}}$ is also smooth and of the rate to be expected. At certain air gaps, however, the behaviour is found to be quite rapidly variable. The frequency changes suddenly become negative, and then positive again within quite narrow ranges of air gap. The values of $\sigma_{1\,\text{min.}}$ also show corresponding fluctuations. These effects, when first noted, strongly suggested further subsidiary resonance phenomena.

Further series of measurements were therefore made with extraordinary care. The micrometer air gap within which the piece of quartz was supported, shown in Fig. 32, p. 203, was found to be uncertain in its action owing to lack of parallelism between the plane faces of the electrodes, consequent upon these being not quite perpendicular to the axis of the micrometer screw. This was a difficult matter to correct, as it is not easy to adjust a plane face 3 cms. in diameter to be so nearly perpendicular to the axis of the micrometer screw that the air gap remains parallel to an accuracy of 0.002 mm. over the whole face for any setting. After a good deal of adjustment it was believed that this degree of accuracy was attained and the final series of readings was taken. The readings taken

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were $\sigma_{1\,\text{min.}}$ and frequency, for successive changes of 0.02 mm. in air gap on each side over most of the range, and for changes of 0.01 mm. over certain parts of the range.

The remarkable curves obtained are shown in Figs. 17 and 18. The upper curve gives the variation of frequency with variation in air gap, and the lower curve gives the corresponding changes in $\sigma_{1\,\text{min.}}$. Both curves have been plotted to the same scale of

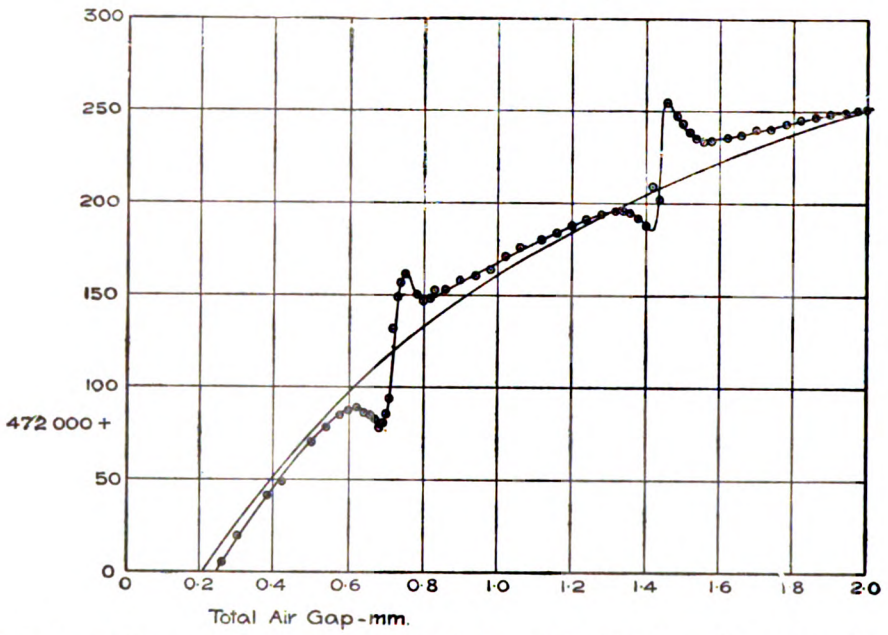

FIG. 17. Variation of Response Frequency with Variation of Air Gap for a Transverse Resonator.

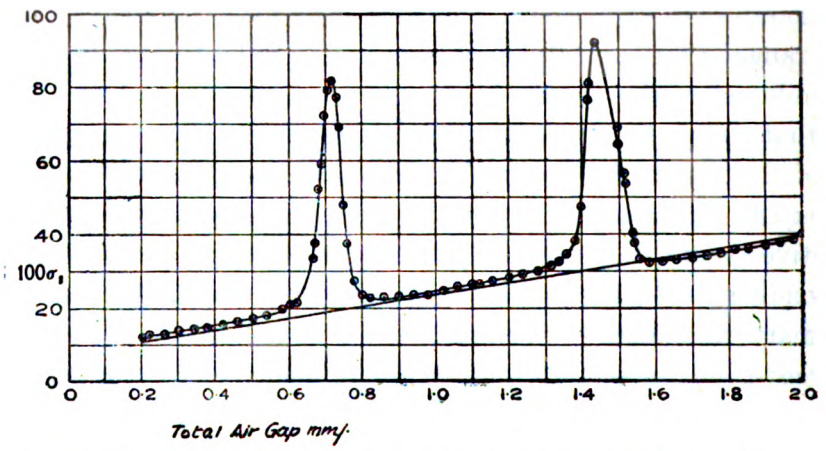

FIG. 18. Variation of Damping Coefficient corresponding to Fig. 17.

air gap and not to a scale of $1/(K_1 + K_2)$ as in Fig. 16. The scale represents total air gap, *i.e.*, a reading of 1.0 mm. means that the gap was 0.5 mm. on each side of the resonator. In order to obtain these curves the utmost care was necessary to ensure that the two air gaps were precisely equal. The equality necessary was certainly to less than 0.01 mm., and was difficult to obtain.

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It will be seen that a large increase in the apparent damping and a corresponding fluctuation in the frequency of $\sigma_{1\,\text{min.}}$ occur at air gaps of 0.72 and 1.44 mm., representing gaps on each side of 0.36 and 0.72 mm. respectively. Now the shape of the two effects suggests immediately to those familiar with coupled circuits that the equivalent of a coupled resonant circuit is operating. There is no doubt that this is the case and that the effect is due to resonance of the plate of air between the surface of the quartz and the electrode. The effect recurs every time the air gap has a value which is an integral multiple of the half wave-length of the supersonic air wave corresponding to the frequency of the resonator. Thus, assuming the value 340 metres per second to hold for the velocity of sound in air in the case of such high-frequency air waves, then, at a frequency of 472,000 vibrations per second the half wave-length is $340{,}000 \div (2 \times 472{,}000) = 0.360$ mm., a result in exact accordance with the observed value.

The damping is enormously increased as a result of the greatly augmented air vibrations occurring in the space between the surface of the quartz and the electrode. The effective value of ϕ_2 is increased twenty-fold owing to this large absorption of energy from the quartz. Whether the energy is wasted as viscous damping in the air, or whether it is radiated as an augmented air wave, the writer is at present unable to say.

The equivalent electrical system will be that of a closed oscillatory circuit coupled inductively to the fictitious inductance N of Fig. 2.

The thin lines on Figs. 17 and 18 are the values of n and $\sigma_{1\,\text{min.}}$ respectively which are obtained by calculation, using the previously determined values of S, K and K_1. Except for the regions where the air resonance introduces its reaction, the agreement is quite good in view of the difficulty of the experiments.

It will be seen on reference to the scale of frequency in Fig. 17 that the change in frequency consequent upon a change in air gap is much smaller for this transverse resonator than for the longitudinal resonator previously given. The total possible change from zero to an infinite air gap is less than one part in a thousand. The horizontal lines of Fig. 17 are spaced roughly one part in ten thousand in frequency apart. On the scale upon which these results are plotted, therefore, a change of one part in a hundred thousand in frequency is easily seen.

Within close limits, therefore, it may be provisionally stated that the behaviour of a transverse resonator can be imitated closely by an equivalent electrical network, but that at certain values of air gap the resonator behaves as though an additional resonant circuit were coupled to the network representing the resonator. This resonant circuit must be considered to have a frequency so varied as to be proportional to K_2 and to become equal to that of the resonator at those values of the air gap which represent integral half wave-lengths in air of the frequency of the resonator.

Current in the Resonator Circuit.—The whole of the preceding considerations refer to the effects upon an oscillatory circuit when the resonator is connected in parallel with the condenser forming the capacity of an oscillatory circuit. It has been shown how an analysis of the network may be made from observations of the current in this circuit.

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This procedure, though much favoured by the writer, is by no means the only method by which the equivalent constants of the quartz resonator may be obtained. We have still to consider the current taken by the resonator itself. This is the current I_2 of Fig. 3 (*b*). But there are circuits to which the resonator may be connected other than the oscillatory circuit that we have been considering.

It has already been shown that the circuit representing the resonator itself is one in which all the parts have very high impedance. Thus, we have K of the order 0.05 $\mu\mu$F for a resonator of frequency of the order 40,000 cycles per second, so that $1/K\omega$ is no less than 80×10^6 ohms. K_2 is of the order 5$\mu\mu$F giving an impedance of about 1 megohm. Measurements on such circuits are by no means easy since the currents are very small except at resonance. It is necessary to use a current-measuring device of high resistance in order to obtain useful deflections. The presence of this resistance in the circuit has a considerable effect in reducing the current at resonance, and increases the apparent decrement of the resonator considerably. By the use of a Duddell thermo-galvanometer with heater of about 100 ohms, it is, however, possible to get observations without risk of breaking the quartz or of vitiating the results owing to reaction on the source. Difficulties also arise owing to parasitic currents to earth, since earth capacities are by no means small compared with K, K_1 and K_2.

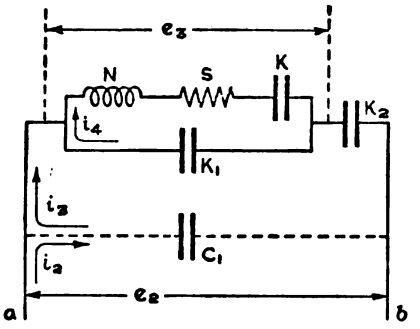

FIG. 19. Diagram of the Equivalent Network of a Resonator connected to a Condenser.

These effects can, however, all be dealt with by careful attention to the circuit arrangements and by a correct procedure in the treatment of the observations.

The cases which may be considered are as follows :—

(1) Constant voltage applied to the terminals of the resonator. Two possibilities arise.

(*a*) The capacity C_1 of Fig. 19 is small but not negligible compared with K_2. Such a capacity always exists in practice due to leads and earth capacities.

(*b*) The capacity C_1 is purposely made large by the use of a condenser, but the voltage e_2 is still considered constant (the use of an inducing coil of a very few turns closely coupled to the source enables this condition to be satisfied).

The second case is :—

(2) The voltage e_2 corresponds to that operating when the resonator is shunted across an approximately tuned resonant circuit.

This is the case used experimentally by CADY, and is complicated to deal with algebraically.

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Cases (1*a*) and (1*b*) will be taken first. The circuit and conditions are as in Fig. 19. The voltage e_2 is to be considered constant. The instantaneous current i_2 is that of which the r.m.s. value I_2 is measured by the thermo-galvanometer. The current i_3 is to be considered inaccessible to measurement, but its r.m.s. value is a real quantity; i_4 has no existence as a current, but has its counterpart in the mechanical vibration of the resonator.

By connexion of the points *a*, *b*, to a coil of small self-inductance and small resistance, so that no approach to resonance occurs in the circuit carrying current i_2, the E.M.F. induced in this coil when coupled to a source will remain sensibly constant over the narrow range of frequency concerned. Experiments under such conditions show that as frequency is smoothly varied, I_2 rises to a sharply defined maximum then falls to a less well-defined minimum, and finally approaches asymptotically its original value. The curve is shown in Fig. 20. It is convenient, from an experimental standpoint, when

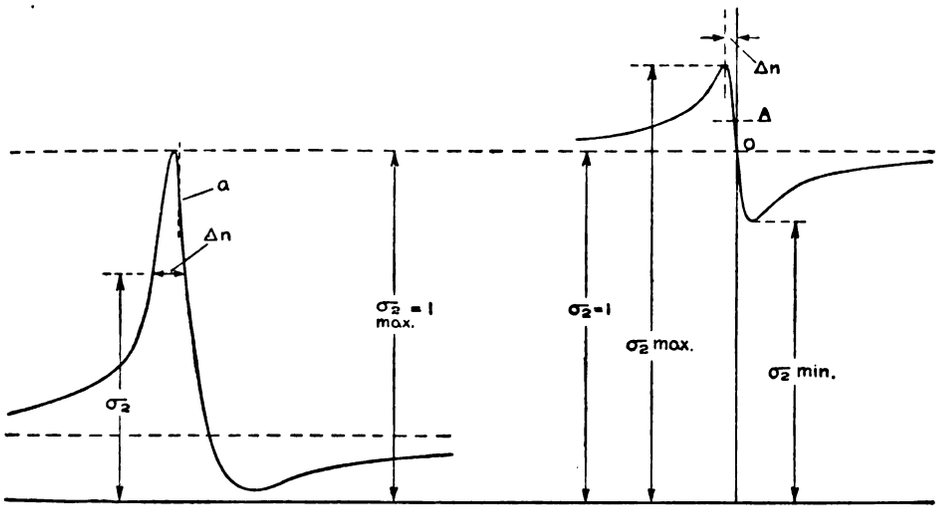

FIG. 20. Current Response Frequency Curve for a Resonator with a small Parasitic Capacity in Parallel.

FIG. 21. As Fig. 20, but with a relatively large Capacity in Parallel.

C_1 is small, to allow the maximum current obtained to be equal to unity, and to call the current ratio at any other point σ_2, where $\sigma_2 = I_2/I_{2\,max.}$.

When C_1 is large, the curve of σ_2 takes on a more symmetrical form of the type shown in Fig. 21. In this case it is more convenient to choose the scale of deflections so that the value of σ_2 approaches unity outside the range where σ_2 changes rapidly. The maximum and minimum values of σ_2 are then connected by the relation $\sigma_{2\,max.} \times \sigma_{2\,min.} = 1$. The two equations for σ_2 are as follows :—

(*a*) C_1 small and $\sigma_{2\,max.}$ chosen equal to unity.

$$\sigma_2^2 = \frac{\phi_2^2 + \left[q + \frac{K(C_1 + K_2)}{C_1K_t + K_1K_2}\right]^2}{\left[\phi_2^2 + \left(q + \frac{K}{K_t}\right)^2\right]\left[1 + \frac{K^2K_2^4}{K_t^2(C_1K_t + K_1K_2)\phi_2^2}\right]} \quad \ldots\ldots \quad (22)$$

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(*b*) C_1 not small, and $\sigma_{2\,max.} \times \sigma_{2\,min.} = 1$.

$$\sigma_2^2 = \frac{\phi_2^2 + \left[q + \frac{K(C_1 + K_2)}{C_1K_t + K_1K_2}\right]^2}{\phi_2^2 + (q + K/K_t)^2} \quad \ldots\ldots\ldots\ldots \quad (23)$$

For certain purposes it is convenient to consider as origin of frequency that value where $q = -K/K_t$, and to write $q = -\frac{K}{K_t} - \frac{2dn}{n}$.

Equation (22) becomes

$$\sigma_2^2 = \frac{n^2\phi_2^2/4 + \left[\frac{nKK_2^2}{2K_t(C_1K_t + K_1K_2)} - dn\right]^2}{\left(\frac{n^2\phi_2^2}{4} + dn^2\right)\left[1 + \frac{K^2K_2^4}{\phi_2^2K_t^2(C_1K_t + K_1K_2)^2}\right]} \quad \ldots\ldots\ldots \quad (24)$$

In case (*b*), however, it is more convenient to choose an origin of frequency such that q is the mean of $\frac{K(C_1 + K_2)}{C_1K_t + K_1K_2}$ and $\frac{K}{K_t}$. We substitute for q, therefore, the quantity

$$q = -\frac{K}{2}\left[\frac{(C_1 + K_2)}{C_1K_t + K_1K_2} + \frac{1}{K_t}\right] - \frac{2dn}{n}.$$

The equation for σ_2 then takes the very convenient form

$$\sigma_2^2 = \frac{\frac{n^2\phi_2^2}{4} + (D - dn)^2}{\frac{n^2\phi_2^2}{4} + (D + dn)^2}, \quad \ldots\ldots\ldots\ldots \quad (25)$$

where

$$D = \frac{nKK_2^2}{4K_t(C_1K_t + K_1K_2)}. \quad \ldots\ldots\ldots\ldots \quad (25a)$$

These equations differ from those considered earlier in the paper in that the point of resonance is not immediately obvious by an inspection of the curves. If we define resonance by the condition that the current I_3 actually flowing in the quartz circuit is in phase with the voltage applied, then σ_2 corresponding to this frequency may have values varying from the point *a* on Fig. 20 very near $\sigma_{2\,max.}$ for the case when C_1 is small, to the point A on Fig. 21 very near $\sigma_2 = 1$ for the case when C_1 is large.

We will consider only equation (25) in developing the means of obtaining ϕ_2 and D. Differentiating (25) with respect to dn for maximum and minimum values of σ_2, it will be found that $\sigma_{2\,max.}$ and Δn at which it occurs (*see* Fig. 21) are connected by the equation

$$(\sigma_{2\,max.})^2 = \frac{2\Delta n/n + \sqrt{4\Delta n^2/n^2 - \phi_2^2}}{2\Delta n/n - \sqrt{4\Delta n^2/n^2 - \phi_2^2}}. \quad \ldots\ldots\ldots \quad (26)$$

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From this equation it is found that

$$\phi_2 = \frac{4\Delta n}{n} \cdot \frac{\sigma_{2\,\max.}}{\sigma_2{}^2{}_{\max.} + 1} \quad \ldots\ldots\ldots\ldots \quad (27)$$

By substitution of ϕ_2 and $\sigma_{2\,\max.}$ into equation (25) we find that

$$D = \frac{2\Delta n}{n} \cdot \frac{\sigma_2{}^2{}_{\max.} - 1}{\sigma_2{}^2{}_{\max.} + 1} \quad \ldots\ldots\ldots\ldots \quad (28)$$

also

$$D = \frac{\phi_2 (\sigma_2{}^2{}_{\max.} - 1)}{2\sigma_{2\,\max.}} \quad \ldots\ldots\ldots\ldots \quad (29)$$

These equations enable ϕ_2 and the quantity A to be determined from the curve of σ_2.

The determination of K and K_1 can be made by observing a family of curves of σ_2 each corresponding to a chosen air gap, and hence to a known value of K_2. From these curves values of A as a function of K_2 may be deduced. This cannot be done very conveniently for case (*a*) in which C_1 is comparable with K_1, since the base line value of σ_2 is too small to be observed with accuracy. All the curves of σ_2 for various values of K_2 will be nearly similar.

In the case (*b*), however, where C_1 is large compared with K_1, we can write equation (25*a*) for D in the form

$$\frac{1}{\sqrt{D}} = \sqrt{\frac{2C_1}{K}}\left(\frac{K_1}{K_2} + 1 + \frac{K_1}{2C_1}\right). \quad \ldots\ldots\ldots \quad (30)$$

If, therefore, we plot the experimentally determined values of $1/\sqrt{D}$ obtained by (29) against $1/K_2$ it is seen from equation (30) that a straight line should result, from which K and K_1 may be observed. At the point $1/K_2 = 0$ the intercept $1/\sqrt{D}$ gives approximately $\sqrt{2C_1/K}$, whence K is determined, since C_1 is known. At the point $1/\sqrt{D} = 0$ we have

$$\frac{1}{K_1} = -\left(\frac{1}{K_2} + \frac{1}{2C_1}\right),$$

giving K_1, and so analysing the resonator mesh. The measurement of ϕ_2 by observation of Δn is not very accurate, since this quantity is small and not too well-defined. Also the determination of ϕ_2 by this relationship only makes use of one point on the curve of σ_2. The curve of σ_2 does not lend itself quite so easily to the development of a straight line of slope proportional to ϕ_2 in the manner already shown for σ_1.

In order to develop such an equation it is convenient and instructive to resort to the vector admittance diagram of the case under consideration, *i.e.*, with the resonator

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system connected to a condenser across which a constant voltage is maintained. Fig. 22 gives the vector admittance and conductance relationship for the case concerned. We have already seen that the vector diagram of the simple quartz mesh given by equations (1) and (2) is a circle, part of which has been given in Fig. 4. Suppose, for the moment, that this impedance circle is that partly shown in Fig. 22 by the dotted arc passing through O_2 and drawn with O_2X_2 as the axis of effective resistance and O_2Y as axis of effective inductance. We set back O_2O_1 along O_2Y equal to $1/K_2\omega$ and draw an axis, O_1X_1, cutting this arc in R. Then O_1R is the effective impedance of the mesh, including K_2 at the frequency of maximum response. Now let us change our scales from those of impedance to those of admittance. The effective impedance O_1R is almost exactly equal to the effective resistance S_1 of Fig. 4, and will become an effective conductance.

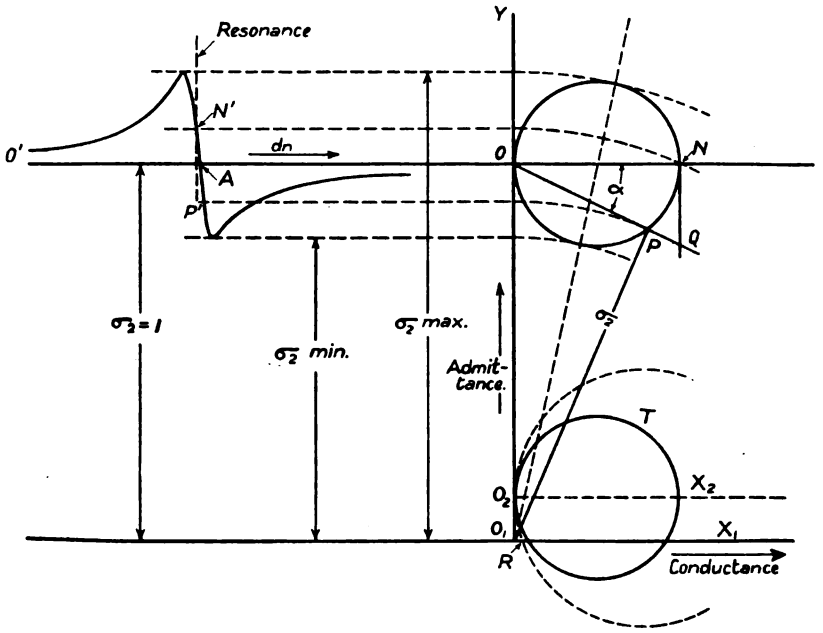

FIG. 22. Graphical Solution for the Curves of Figs. 20 and 21.

We can choose the scales so that O_2O_1 (Fig. 22) now becomes the permittance $K_2\omega$. It is a property of the circular impedance diagram that its inverse admittance diagram is also a circle. This circle is shown as O_2TX_2 of diameter equal to $1/O_1R$. We now set out a new axis ON, such that OO_2 is equal to the permittance $C_1\omega$ of the condenser C_1 of Fig. 6. The circle O_2TX_2 is then redrawn at ONP. The construction of the current-frequency curve O'N'P follows along lines somewhat similar to those already clearly set out by MALLETT.* In the present case the circle is tangential at O. Any vector admittance O_1P is now shown correctly as regards phase angle and magnitude with respect to OO_1 for a value of σ_2 corresponding to a point P' on the curve O'N'P' at any chosen frequency difference *dn*.

* E. MALLETT and A. D. BLUMLEIN, 'J. Inst. Elec. Eng.,' Vol. 63, p. 397, 1925; E. MALLETT and V. J. TERRY, 'Wireless World,' Vol. 16, p. 630, 1925.

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Any point P′ on the curve O′N′P′ of σ_2 has its corresponding point P on the circle ONP. MALLETT has shown that the tangent of the angle $\alpha = \text{PON}$ is proportional to the difference of the frequencies to which P and N (the resonant frequency) correspond. (The circle is almost completely described for a frequency change of one part in a thousand.)

If we calculate the value of $\tan \alpha$ in terms of σ_2 and $\sigma_{2\,\text{max.}}$ by simple trigonometry from Fig. 22, it will be found for the case in which we have allowed $\sigma_{2\,\text{max.}} \times \sigma_{2\,\text{min.}}$ to be unity (Fig. 22) that $\tan \alpha$ is given by the expression

$$1 + \tan^2 \alpha = \frac{\left(\sigma_{2\,\text{max.}} + \frac{1}{\sigma_{2\,\text{max.}}}\right)^2}{1 + \sigma_2^2 \pm \frac{2\sigma_{2\,\text{max.}}}{\sigma_{2\,\text{max.}}^2 - 1} \sqrt{(\sigma_{2\,\text{max.}}^2 - \sigma_2^2)\left(\sigma_2^2 - \frac{1}{\sigma_{2\,\text{max.}}^2}\right)}}, \quad \ldots \quad (31)$$

an expression not containing dn. For the case where C_1 is small and $\sigma_{2\,\text{max.}}$ is chosen equal to unity, the equation becomes

$$1 + \tan^2 \alpha = \frac{3 + \sigma_b^4}{1 + \sigma_2^2 \pm \frac{2}{1 - \sigma_b^2} \sqrt{(1 - \sigma_2^2)(\sigma_2^2 - \sigma_b^4)}}, \quad \ldots \quad (32)$$

where σ_b stands for the base line value of σ_2 at frequencies removed from the immediate neighbourhood of response.

Referring back to the circle diagram of Fig. 22, the angle which the admittance vector OP makes with ON is, of course, the same which the corresponding reactance vector on the dotted reactance circle will make with the resistance axis O_2X_2. Hence,

$$\tan \alpha = \frac{1}{S_0 K_0 \omega}$$

if q in equations (1) and (2) is put equal to $-K/K_t - 2dn/n$. Inserting this value of q into (1) and (2) and calculating $S_0 K_0 \omega$ therefrom, we obtain the very simple relationship

$$\tan \alpha = \frac{2dn}{n} \cdot \frac{1}{\phi_2},$$

where dn is measured from the true response frequency.

On plotting $\tan \alpha$ against dn a straight line should be obtained, the equation of which may be written $n = m \tan \alpha + n_0$, from which $\phi_2 = 2m/n$, and n_0 is the true response frequency, differing very slightly in frequency from the peak in Fig. 20 or from the point O in Fig. 21. In an average satisfactory specimen of quartz this difference is not greater than one part in a hundred thousand. Examples illustrating the cases just considered are shown in Figs. 23 to 26.

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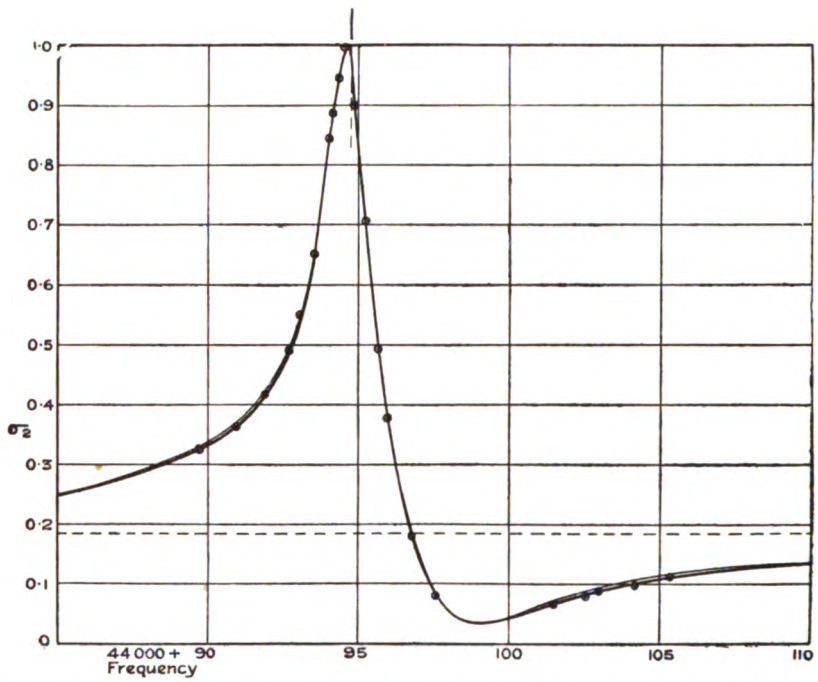

FIG. 23. Experimental and Calculated Curves of the Type shown in Fig. 20.

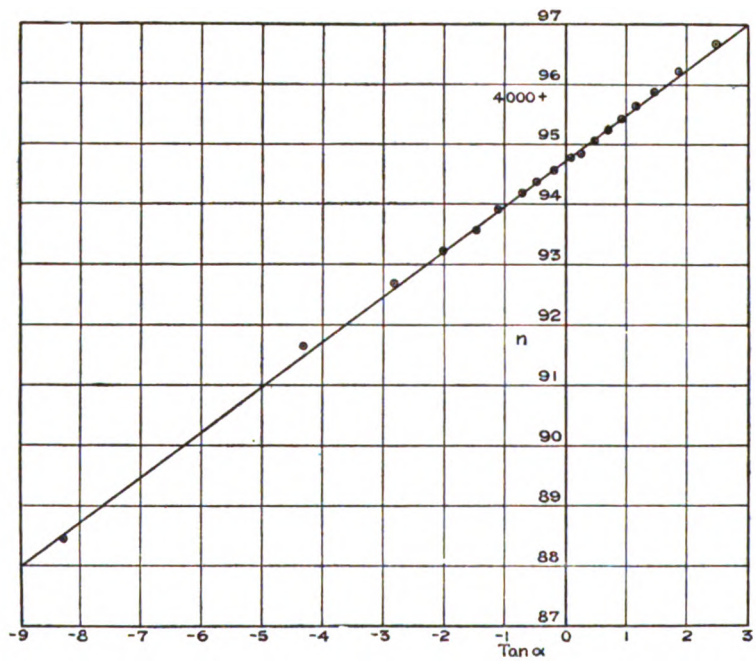

FIG. 24. Linear Relationship deduced from the Experimental Curve of Fig. 23.

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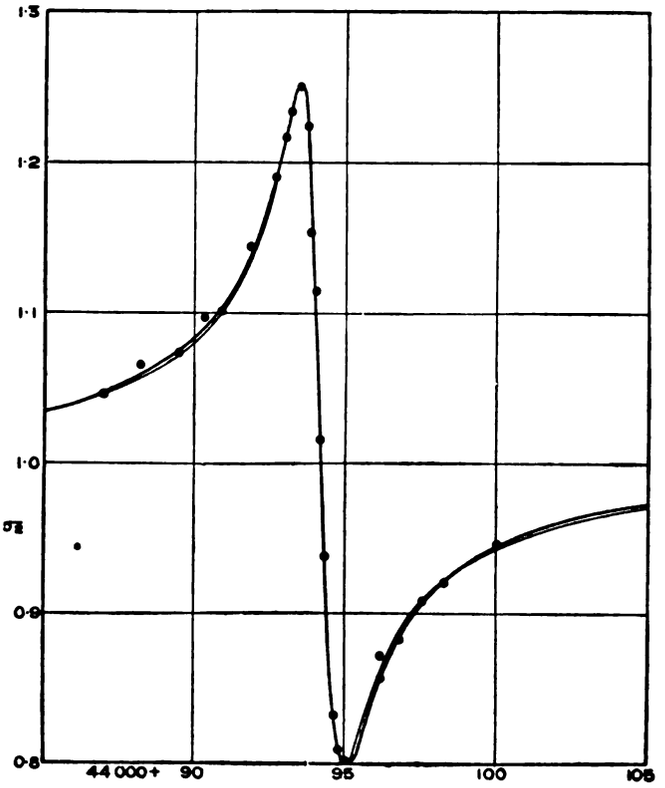

FIG. 25. Experimental and Calculated Response Curves corresponding to the Case of Fig. 21.

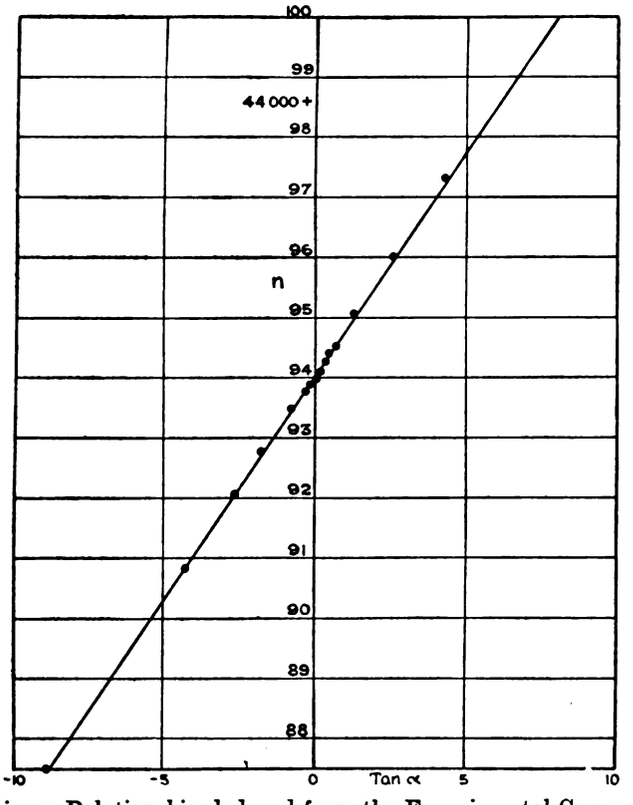

FIG. 26. Linear Relationship deduced from the Experimental Curve of Fig. 25.

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The curves refer to the bar vibrating in its lowest longitudinal mode at a frequency of 44,000 cycles per second. The total air gap was 1 mm. Fig. 23 gives the experimental curve of σ_2 obtained for the case where C_1 represents only the stray and parasitic earth capacities. The curve is drawn as a heavy line and the points observed are shown as dots in circles.

From observations of n given by this curve at chosen values of σ_2, and from the calculated values of tan α using equation (32), corresponding values of tan α and n have been plotted in Fig. 24. In view of the difficulty in accurately observing such small frequency differences from the curve it will be seen that the points lie on a fairly good line.

The slope of the line gives the value 34.0×10^{-6} for ϕ_2, which agrees very well with values obtained on other occasions by the other method. The response frequency occurs at $44{,}094.7_2$, and is shown by a dotted vertical line on Fig. 23. From this value of ϕ_2, and from the known value of σ_b, the value of D has been calculated, using the equivalent equation to (29). The curve of σ_2 has then been calculated using equation (25) and multiplying the results by the factor necessary to bring $\sigma_{2\,\text{min.}}$ to unity. The curve is shown as a fine line on Fig. 23. The agreement between the observed and the calculated curves is very good. Figs. 25 and 26 show the case of the same bar in the same gap, but the condenser C_1 has been made equal to 350 $\mu\mu$F. The line of Fig. 26 is fairly good, and the value deduced for the slope, 33.7×10^{-6}, is in very good agreement with the previous case. The agreement between observed and calculated curves in Fig. 25 is again remarkably good. It will be noticed that the frequency $n_0 = 44{,}094.06$. There is a temperature coefficient of $+ 12 \times 10^{-6}$ in the tuning-fork multivibrator, in terms of which the frequencies were measured. The temperature difference of the fork for the two sets of measurements was probably 1° C. in a direction such that the second value, 44,094.06, when corrected to the temperature of the first measurement, becomes probably $44{,}094.5_4$. The two frequencies differ, therefore, by only four parts in a million, which, in view of the fact that the two tests were made on different days, is a very satisfactory agreement.

The further analysis of the resonator by the use of various air gaps has not been carried out, but there is no reason to suppose that complete agreement with the results obtained by the other method would not be obtained. Curves similar to those of Figs. 23 and 25 have been obtained for other sizes of bar and for a plate vibrating transversely.

These experiments, therefore, uphold the equivalent network theory of the resonator, and are in complete accord with those made in the tuned circuit case.

Before bringing this part of the paper to a conclusion, the case (2) must briefly be considered. In this case the current in the branch circuit containing the resonator is measured under the circuit conditions of Fig. 5, in which the resonator is shunted across an oscillatory circuit approximately tuned to the response frequency of the resonator. If, from the original equations 1, 2, 3, we solve for I_2 in terms of the value to which it

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tends at frequencies near but outside the belt of response, we can express this as a ratio ρ_2 similar to that which has already been considered as σ_2.

The general equation for ρ_2 becomes

$$\rho_2^2 = \phi_1^2 \frac{\phi_2^2 + \left[q + \frac{K(C_1 + K_2)}{C_1K_t + K_1K_2}\right]^2}{\left[pq + \frac{pK(C + C_1 + K_2) - q(C_1K_t + K_1K_2) - K(C_1 + K_t)}{CK_t + C_1K_t + K_1K_2} - \phi_1\phi_2\right]^2 + \left[q\phi_1 + p\phi_2 + \frac{\phi_1K(C + C_1 + K_2) - \phi_2(C_1K_t + K_1K_2)}{CK_t + C_1K_t + K_1K_2}\right]^2}; \quad (33)$$

where all the terms have exactly the same meanings as in previous equations. C_1 is a small parasitic capacity shunting the resonator itself, and includes the capacity of the electrodes outside the quartz, and of the leads to them. This capacity cannot be eliminated, and we cannot measure the true current flowing into the resonator itself.

In the original equation (4) for σ_1 we have not considered the effect of C_1. We are there dealing with the large current flowing through the inductance L. The effect of C_1 upon this is comparatively small, since C_1 is merely a small capacity in parallel with C. If the full expression for σ_1 is worked out, including the C_1 term, it is found that, in order to express σ_1 in terms of the frequency difference from the bottom of the crevasse, it is necessary to substitute for q the quantity

$$q = -\frac{K(C + C_1 + K_2)}{CK_t + C_1K_t + K_1K_2} - \frac{2dn}{n}.$$

If this quantity is substituted into the equation for ρ_2 we get

Case I.—C = constant.

$$[C \text{ const.}]\ \rho_2^2 = \frac{A^2 + (B - dn)^2}{\left[\frac{\phi_2}{A\phi_1}dn^2 + dn\left(\frac{D}{B} + \frac{2\Delta n}{n\phi_1}\right) - (D + A)\right]^2 + \left[dn(1 + \phi_2/\phi_1) + A\left(\frac{D}{B} + \frac{2\Delta n}{n\phi_1}\right)\right]^2}, \quad \ldots \quad (34)$$

in which the following substitutions have been made :—

$$A = \frac{n\phi_2}{2}; \quad B = \frac{nKCK_2^2}{2(C_1K_t + K_1K_2)(CK_t + C_1K_t + K_1K_2)};$$

$$D = \frac{nKCK_2^2}{2\phi_1(CK_t + C_1K_t + K_1K_2)^2}.$$

It is to be noted that B and D have different meanings here from those used earlier in the paper. In this equation C is constant, and may have any value whatever. The quantity p in such a case is best substituted by an expression of the form

$$p = -\frac{2\Delta n}{n} - \frac{2dn}{n},$$

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where Δn is a constant frequency difference (generally small), depending upon the setting of C. It will be remembered that in equation (7*a*) for σ_1 it was noted that the equivalent term to $D/B + 2\Delta n/n\phi_1$ determined the asymmetry of the σ_1 curve.

Case II.—LC circuit adjusted so that at each value of dn, σ_1 is made a maximum.

This case, in which C is varied in order that σ_1 may be a maximum, as in CADY's experiments, is complicated to work out accurately, since C appears in so many places in equation (5) for σ_1^2 when the quantity $(1 - LC\omega^2)$ is substituted for p. But in many cases we may consider ϕ_1 and D constant as long as the changes in C are not more than a few per cent.

Under these conditions the only quantity which need be considered as variable, when C is varied, is the quantity p.

The conditions for σ_1 to be a maximum with respect to variation of C are then reduced to a differentiation of the equation for σ_1^2 with respect to p as the variable.

Writing equation (5) in the form

$$\sigma_1^2 = \frac{A^2 + dn^2}{\left[-\frac{dn}{\phi_1}p + \frac{D}{B}dn - (D + A)\right]^2 + \left[\frac{A}{\phi_1}p - dn - \frac{AD}{B}\right]^2}, \quad \ldots \quad (35)$$

and differentiating with respect to p it will be found that for σ_1 to be a maximum p must have the value

$$p = \frac{D\phi_1}{B} - D\phi_1 \frac{dn}{A^2 + dn^2} \quad \ldots \quad (36)$$

The equations for σ_1 and ρ_2 become, by substitution of p as in (36),

$$\sigma_{1\,max.} = \frac{A^2 + dn^2}{AD + A^2 + dn^2} \quad \ldots \quad (37)$$

$$\rho_{2\,max.}^2 = \frac{(A^2 + dn^2)\cdot[A^2 + (B - dn)^2]}{(AD + A^2 + dn^2)^2} \quad \ldots \quad (38)$$

Although we are strictly investigating ρ_2, and not σ_1 when σ_1 is a maximum, the equation for σ_1 is here given. It is seen to represent a curve symmetrical with respect to dn, and, unlike the previous equation for σ_1, equation (37) gives a curve rising asymptotically to unity and never falling again.

Equation (38) has been called $\rho_2^2{}_{max.}$ This is not strictly true when the variation of C is such as to make σ_1 a maximum when the fully accurate case is taken, because C appears in the numerator in the general equation for ρ_2 but is absent from that for σ_1. For all practical purposes, however, the term B in equation (38) may be considered independent of C so long as C is large compared with C_1 and K_1.

This equation represents the peculiar curve obtained experimentally by CADY. It is seen that when $dn = 0$ a sharp minimum occurs on account of the bracket $A^2 + dn^2$,

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also when $dn = +$ B another minimum occurs. Two maxima occur at values of dn near zero, one positive and the other negative (Fig 27*b*).

It remains only to consider the changes in C necessary to satisfy the equation for p.

We may write for p the expression

$$-p = \frac{2dn}{n_0} + \frac{dc}{C_0}, \quad \ldots \ldots \ldots \ldots \ldots \quad (39)$$

where n_0 and C_0 are constant and are related by the equation $4\pi^2 n_0^2 L C_0 = 1$, and n_0 is the frequency at which $dn = 0$.

If this value of p is substituted into equation (36), and if we write

$$\frac{\Delta C}{C_0} = \frac{dc}{C_0} + \frac{D\phi_1}{B},$$

in order that capacity changes may be measured from the point $dn = 0$, it will be found that

$$\frac{\Delta C}{C_0} = D\phi_1 \frac{dn}{A^2 + dn^2} - \frac{2dn}{n_0}. \quad \ldots \ldots \ldots \quad (40)$$

This equation gives the change in capacity necessary to make a maximum at each chosen value of dn.

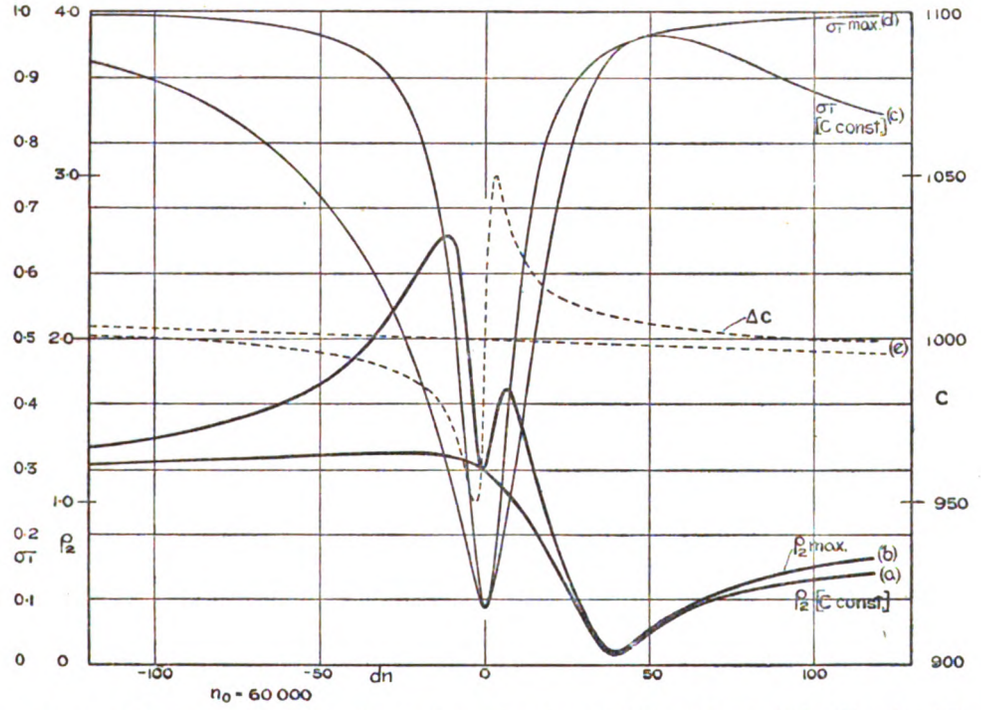

FIG. 27. Group of Curves of σ_1 and ρ_2 for various Experimental Conditions of the Circuit of Fig. 5.

As examples of these equations, the set of curves shown in Fig. 27 have been calculated for the following assumed case :—

$$n_0 = 60{,}000, \qquad A = \frac{n\phi_2}{2} = 3, \qquad \phi_2 = 100 \times 10^{-6}$$

$$D = 30.33, \qquad B = 40, \qquad \phi_1 = 1 \times 10^{-2}, \qquad \Delta n = -77.5.$$

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The last term does not enter into the equations for $\sigma_{1\,max.}$ or $\rho_{2\,max.}$, but determines the asymmetry of σ_1. It has been given the arbitrary value of — 77.5 in order to make the curve for σ_1 reasonably but not too unsymmetrical.

The curves which have been drawn are as follows :—

(*a*) ρ_2 when C is constant (equation 33).

(*b*) ρ_2 when C is adjusted so that σ_1 is a maximum (equation 38).

(*c*) σ_1 when C is constant (equation 7*a*).

(*d*) $\sigma_{1\,max.}$ (equation 37).

(*e*) ΔC, to satisfy equation (40), using a base line value of 1,000 $\mu\mu$F when $dn = 0$.

The similarity of curves (*b*), (*d*) and (*e*) to those observed by CADY will be noted, and shows that the simple algebraic method of considering the electrical system is substantially in agreement with observation. A curve similar to (*a*) for a case where C was constant was observed, and its form was entirely in agreement with that calculated for the above fictitious case.

The equations given for $\phi_{2\,max.}$ and $\sigma_{1\,max.}$ are only approximate when ΔC is, say, 10 to 20 per cent. of C_0, since all the terms containing C will be considerably changed. If, however, the alternative procedure of changing L were adopted, the equations would become exact, and further deductions from them could be made in order to use this means of analysing the resonator mesh. The curve (*e*) would then, of course, be a curve of ΔL, and would be exactly the same in sense and proportions as that for ΔC.

In the example chosen the values of A, B and D are met by choosing the following reasonable electrical quantities :—

$$K_1 = K_2 = C_1 = 5\ \mu\mu F.$$

$$K = 0.04\ \mu\mu F \text{ and } C_0 \doteqdot 1{,}000\ \mu\mu F.$$

The large effect on ρ_2 of the parasitic capacity (C_1) of 5 $\mu\mu$F is seen by consideration of B.

If C_1 is neglected

$$B \doteqdot \frac{nKK_2}{2K_1K_t} \doteqdot 120 \text{ cycles per second.}$$

Including C_1 at the assumed value of 5 $\mu\mu$F,

$$B \doteqdot 40 \text{ cycles per second.}$$

Since B represents the frequency difference between the two minimum values of $\rho_{2\,max.}$, and since also the value of $\rho_{2\,max.}$ at $dn = 0$ is proportional to B, it is seen that the shape of the curve is very largely dependent upon C_1.

From the equations for $\rho_{2\,max.}$ a series of equations can be developed for the purpose of evaluating ϕ_2, K and K_1 when using such a method.

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Thus ϕ_2 becomes given by the equation

$$\phi_2 = \frac{\delta n}{n} \sqrt{\frac{\text{max. } \sigma_{1\,\text{min.}} (1 - \sigma_{1\,\text{max.}})}{\sigma_{1\,\text{max.}} - \text{max. } \sigma_{1\,\text{min.}}}},$$

where δn is the frequency difference across the crevasse at a chosen value of $\sigma_{1\,\text{max.}}$. Other equations enable D to be obtained for various values of K_2 and ϕ_1, whereby K and K_1 are determined exactly as already developed.

Experimental.

A great number of resonance curves have been made—two hundred or more—on resonators of all sizes from minute bars only 3 mms. long to bars 120 mms. long and 6 mms. thick ; also on plates and discs from those only 0.2 mm. thick and a few millimetres in diameter, up to a large block 20 mms. thick by 48 mms. by 30 mms. These have been examined not only with respect to their respective fundamental modes of longitudinal and transverse vibration, but also with respect to the overtone modes when the bar is vibrating in 1, 2, 3, etc., up to 9 or more segments. Except in certain very special cases, the electrical equivalent circuit has been found to hold true. The exceptions are those in which two response frequencies are so near together that the crevasses intersect one another.

Numerous measurements, referred to later, have been made on the temperature coefficient of response frequency and on other effects on frequency and damping not included in the theory. A very great deal, however, remains to be done in several directions. The subject is one full of interest both from the scientific and practical points of view.

The following information and brief description of the experimental arrangements adopted may, therefore, be of interest.

Measurement of σ_1 and of Frequency.—The apparatus was arranged as in Fig. 28, and consisted of three parts :—

(i) The source, in the form of a simple valve oscillator of ordinary type, and not of the type in which the frequency is controlled by a master valve feeding the grid of a power valve.

It consists of a 250-watt valve, excited from a 400-volt accumulator battery. The inductance and grid coils L_1 are constructed to remain constant in inductance, and the coil in the oscillatory circuit is of small log. dec. These qualities are important where it is necessary to reduce the drift of frequency with time to the smallest amount.

The main part of the oscillatory circuit capacity C_1 was in the form of mica condensers of the clamped type. These again are very invariable in capacity with respect to temperature. In general, C_1 was at least 15,000 $\mu\mu$F. C_2 consisted of two variable standard air condensers in parallel ; one had a range of about 1,500 $\mu\mu$F and the other was of much smaller range with an open scale on which a change of 0.1 $\mu\mu$F was easily

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observed. Sometimes the open-range condenser was connected across only a few turns of the inductance coil in order to secure greater openness of the frequency scale. All

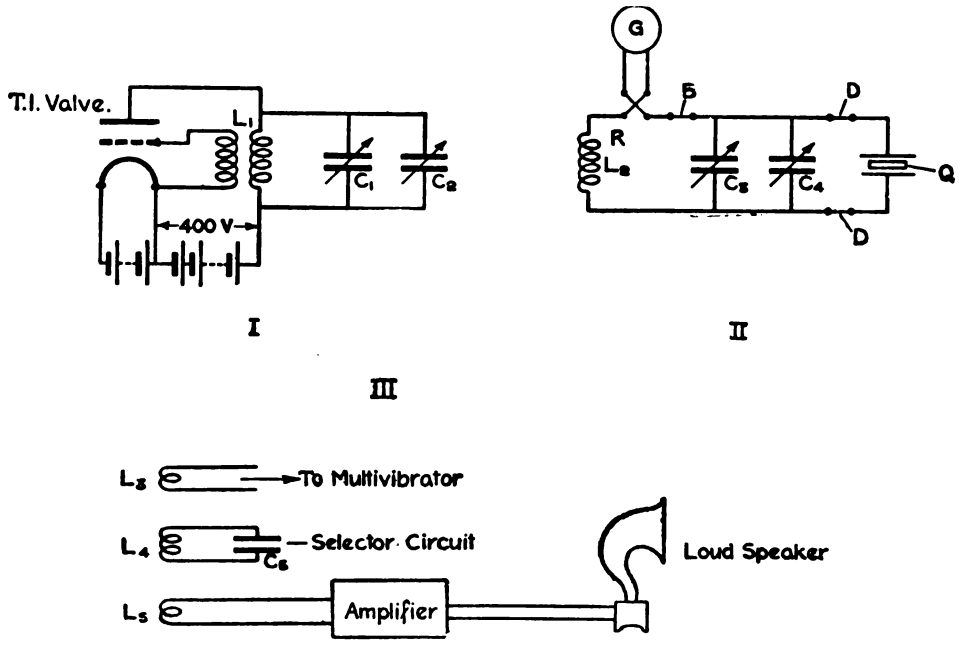

FIG. 28. Arrangement of Apparatus used in the measurement of σ_1 and Frequency.

the condensers were screened in metal cases connected together and to earth. The variable condensers were fitted with long handles to enable precise adjustments to be made from the position at which the galvanometer scale was located.

(ii) The test circuit consisted in general of the oscillatory circuit formed by a standard inductance coil L_2 and a screened variable air condenser C_3, usually shunted by a small open-scale condenser C_4. A vacuum thermo-junction and heater current measurer was inserted where shown and connected to a bifilar galvanometer of good zero-keeping qualities. A small mercury contact link at B served as a switch and as a gap for the insertion of known standard high-frequency resistance units when so desired.

The quartz resonator, mounted in its special gap, was connected by thin wires *via* mercury links DD to the terminals of the condenser C_3. In many cases the resonator was left open to the room, but in others, and particularly in the temperature coefficient measurements, the resonator was placed in a simple electric oven thermostatically controlled in temperature at any desired value up to 50° C. Arrangements were also made to pump dry air into the oven on occasion, in order to ensure that moisture did not condense on the quartz on cooling.

The observations made were of σ_1 at various settings of the condenser C_2 in the source. The preliminary settings on C_3 and C_4 were first made so that the conditions desired in the test circuit were obtained, such as $LC_4\omega^2 = 1$ when $dn = 0$, etc. The coupling was adjusted so that $100\,\sigma_1$ on each side of the crevasse was about 99.8 in

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a normal case, as this was the value found by experience and by calculation to represent the condition $\sigma_0 = 1.000$ in the absence of the resonator.

The adjustments of the condensers C_1 and C_2 were such that C_2 was at about the middle of its scale when σ_1 was at $\sigma_{1\,\mathrm{min.}}$ Readings of σ_1 were then taken at various settings of C_2 suited to the accurate delineation of the curve.

Frequency Measurements.—The arrangements for the measurement of the frequency of $\sigma_{1\,\mathrm{min.}}$ and of the frequency changes corresponding to changes in C_2 were made with very great accuracy by the help of the standard multivibrator wave-meter of the Laboratory.

These arrangements are merely outlined in Fig. 28 (iii), and the reader is referred to the author's original paper* for information regarding the wave-meter itself. It will suffice to say that impulses of exactly known frequency are sent through a coil L_3 which is loosely coupled to a resonant circuit L_4C_5 of small damping coefficient. The condenser of this circuit is variable and is calibrated in terms of harmonics of the frequency of the impulse current through L_3. By setting the condenser C_5 to the reading corresponding to a chosen harmonic, this frequency is reinforced and interferes with that of the source (i). The two frequencies are received by a selective amplifier tuned to resonance, and after rectification and amplification the telephonic frequency beat note is heard in the loud-speaker.

The multivibrator source is controlled by a valve-maintained tuning-fork. The tuning-fork frequency is so constant that it confers upon the whole system of frequency measurement a most remarkable and unique property which is worthy of mention. We have in the wave-meter a system consisting of a valve-maintained tuning-fork of frequency adjusted to 1,000.00 cycles per second. This fork, of elinvar steel, has a temperature coefficient of frequency of + 12 parts in a million per 1° C. rise in temperature. This tuning-fork controls a low-frequency multivibrator of 1,000 impulses per second. All harmonics up to the one hundred and twentieth can be selected from this multivibrator so that a "spectrum" of frequencies up to this limit is obtained by simple selection on the harmonic-selector circuit. Higher frequencies are obtained by the use of a second multivibrator giving impulses at the rate of 20,000 per second. This multivibrator is kept in harmonic synchronism with the low-frequency one by the intermediary of a fixed-frequency resonant circuit set to 20,000 cycles per second, and a single-valve amplifier. By the help of additional coils on the selector circuit, harmonics up to about the seventy-fifth can be selected from this multivibrator.

It will be seen, therefore, that the source can be directly set to any frequency that is an integral number of kilocycles per second up to 120 kc./sec. by the aid of one multivibrator, and to any frequency which is an integral multiple of 20 kc./sec. up to about 1,500 kc./sec. Actually, however, all the intermediate integral kilocycles per second can be successively set and counted round between any two main harmonics such as

* D. W. Dye, 'N.P.L. Collected Researches,' Vol. 18, Paper 13, p. 229 (or 'Phil. Trans. Roy. Soc.,' A, Vol. 224, p. 259, 1924).

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the thirtieth and the thirty-first on the high-frequency multivibrator giving 600 and 620 kc./sec. respectively. Not only can such frequencies be exactly set on the source, but a very great many other subsidiary frequencies may be obtained within the belt of a thousand cycles per second. Thus, whenever the source has a frequency very close to that represented by the formula

$$n = 1{,}000\left(f \pm \frac{1}{r}\right),$$

where f and r are integers, a slow synchronisation beat is heard. For example, in the case of the resonator of 44,000 cycles per second the following frequencies may be exactly set on the source without any auxiliary apparatus: 44,100, 44,111.1, 44,125, 44,142.8, 44,166.7, etc. With care, further intermediary frequencies can be located between these, such as

$$44{,}125 + \frac{142{\cdot}8 - 125}{1,\ 2,\ 3,\ \text{etc.}}$$

This property of the tuning-fork-controlled multivibrator is quite unique and is only possible on account of the extreme steadiness of frequency of the tuning-fork.

Table III opposite gives a typical set of observations as actually taken on a resonator, and shows that it is never necessary to interpolate by more than about 2 cycles per second from a frequency that is known.

The frequencies corresponding to the partial harmonics such as $1{,}000/9.5 = 105.2_5$, etc., have been inserted in the table as accurately-known frequencies self-determined by the wave-meter. The other frequencies are obtained by interpolation from a line drawn through these known frequencies when plotted against condenser reading. It will be seen from the figures that the uncertainties associated with the frequency corresponding to any observed value of C_2 are only of the order 0.1 cycle per second at any part of the curve.

Observations of σ_2.—These were made using a Duddell thermo-galvanometer in which the heater was a thin platinised-quartz rod. The resistance was about 120 ohms and the sensitivity such that full-scale deflection was obtained with a current of about 0.5 milliampere.

In these observations we are dealing with a circuit that is not in resonance but which is almost a pure capacity. As a consequence the voltage which must be induced is much greater than when a resonant test circuit is used. Closer coupling to the source is therefore necessary for a given energy dissipation in such a case. The changes in frequency are confined to such a narrow belt when observing curves such as those of Figs. 24 and 25 that reaction on the frequency of the source may become troublesome. It is for this reason that it is considered highly desirable to use the resonant-circuit method of determining resonant frequency. A disadvantage of the method whereby

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TABLE III.

Condenser reading.	Partial harmonic.	Frequency (cycles per sec.) 44,000+	$100\,\sigma_1$.
50			98·4
53·2	$\frac{1}{9}$	111·1	98·0
60			97·7
65			96·7
70			95·6
75			93·75
78·0	$\frac{1}{9\cdot333}$	107·2	90·9
81			88·4
84			85·2
87			80·2
89·5	$\frac{1}{9\cdot5}$	$105\cdot2_5$	73·2
92			65·7
95			56·6
97			44·4
98			34·6
99			$33\cdot2_5$
100			$33\cdot2_5$
101			34·55
99·5			33·05
102·8	$\frac{1}{9\cdot667}$	103·4	36·4
103			40·8
104			44·6
106			58·8
107·6	$\frac{1}{9\cdot75}$	102·5	58·8
110			67·2
114			77·2
120			86·3
123·5	$\frac{1}{10}$	100·0	89·4
130			93·0
135			94·8
140	$\frac{1}{10\cdot25}$	97·6	96·6
—	$\frac{1}{10\cdot5}$	$95\cdot2_5$	97·7
—	$\frac{1}{11\cdot0}$	90·9	99·1

the constants of any electrical circuit at radio frequencies are determined by the change in effective reactance rather than in effective resistance is that it becomes almost essential to use a source complicated by the addition of a master-valve oscillator to keep the frequency in the power-valve generator independent of load.

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It might be argued that the point of response frequency is defined much more sharply in a curve of the type in Fig. 25 than in one of the crevasse type, owing to the fact that one curve runs through the response frequency at a steep angle, whilst the other is the less well-defined horizontal tangent. In some systems not highly resonant this is true, and for such systems the measurement of resonant frequency is best made by the more accurate method. In the case of a quartz resonator the trouble is, not to determine the resonant frequency, but to avoid missing it altogether owing to its extreme sharpness of location. The accuracy of location is extraordinarily great by the method using a resonant circuit. The variations possible in the response frequency due to other causes in the mounted resonator are far greater than any uncertainty due to the method of observation.

Although in the experiments here described the frequencies have been measured with an elaborate and very accurate standard, it is not at all essential to use such a method for the purpose of investigating the equivalent electrical properties of resonators. With the exception of measurements of temperature coefficient and effects which require a considerable time in their determination, the changes in frequency may be observed by simple calibration of the changes in the source when C_2 is varied, using an auxiliary oscillator of frequency which is sensibly constant during the experiments. A second similar quartz oscillator could also be used, of course.

The *temperature* coefficients of frequency of resonators will now be discussed.

Temperature Coefficients of Frequency.—The temperature coefficient of any standard is frequently of vital importance, and in view of the precision obtainable in other respects the variations with temperature of the response frequency of resonators should be known. Measurements of this quantity were therefore made on a considerable number of resonators and on three oscillators.

Considerable trouble was experienced in the measurements at first owing to the concurrent variations of the air gap with temperature masking the true changes in the response frequency. These were overcome by the use of suitable mountings for the

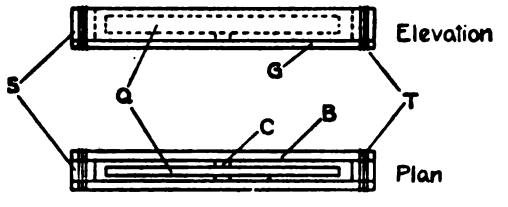

FIG. 29. Holder for a Longitudinal Resonator for Temp. Coefficient of Frequency Measurements.

electrodes and the careful adjustment of the resonator between them. For the longitudinal resonators, the form of the electrodes and the mounting of the resonator were as shown in plan and elevation in Fig. 29.

The electrodes are the narrow brass plates B having their inner faces smooth and flat. These are kept at an accurately known and constant distance apart by small distance

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pieces of fused silica S, having plane parallel faces. The base consists of a narrow glass slip G, having a small piece of cork C stuck on transversely at the centre of its upper surface. The electrodes with their separators and the glass slip are bound together at each end by silk thread. A narrow trough is thus formed within which the resonator is centrally balanced on the narrow cork support. Using this form of mounting remarkably consistent results could be obtained notwithstanding the fact that in some cases the temperature coefficient was only 2 or 3 parts in a million per 1° C.

The arrangement for the transverse resonators was the same in principle as that described above, but disc or rectangular shaped electrodes were used. The lower electrode as shown in Fig. 30 was horizontal. Upon this the resonator was rested on three minute pieces of cork sandpapered to the correct thickness. Three pieces of fused silica of equal thickness with parallel faces served to support the upper electrode.

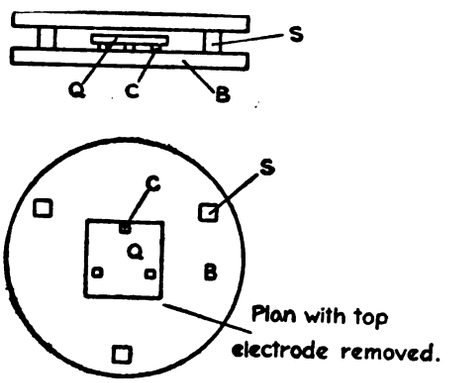

FIG. 30. As Fig. 29, but for a Transverse Resonator.

The mounted specimens were placed in a simple temperature-controlled oven, and the response frequencies were measured over a range of temperature from about 12° C. to about 40° C. Much higher temperatures could have been explored if desired, but for the purpose in view the range given was considered sufficient. Several longitudinal resonators of various lengths and shapes were measured, and three or four transverse resonators were also observed. In the use of some of the longitudinal resonators various overtone modes of vibration were measured, since these gave results of considerable interest. Some complete mounted resonators as made by Messrs. Adam Hilger, Ltd. were also tested, and the author is indebted to them for permission to include these results.

The bars measured had the dimensions given in Table IV below :—

TABLE IV.[1]

Dimensions in millimetres.

Designation.	Length.	Thickness.	Breadth (along optic axis).
	Longitudinal Resonators.		
A	47·0	3·3	7·50
B	62·2	1·52	7·54
C	62·5	1·49	7·50
D	9·86	6·72	7·54
E	Hilger resonators—Dimensions unknown.		
F	,,	,,	
	Transverse Resonators.		
G	19·18	2·71	14·61
H	16·42	6·15	14·65
I (disc)	36 mms. diam., 6 mms. thick.		

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The results obtained on these bars were as follows :—

TABLE V.—Longitudinal Resonators.

Bar.	Mode of vibration.	Frequency, cycles per second.	Temp. coeff., parts in a million.	$\phi_2 \times 10^6$.
A	Fundamental	58,675	$-1 \cdot 0_3$	75
	Overtone in 3 segments	174,450	$-3 \cdot 0_0$	—
	,, 5 ,, ...	273,750	−49	—
	,, 7 ,, ...	414,000	−70	—
B	Fundamental	44,120	$-4 \cdot 0_0$	27
	Overtone in 3 segments	130,850	$-15 \cdot 0_3$	55
	,, 5 ,, ...	219,840	$-5 \cdot 0_2$	19·5
	,, 7 ,, ...	304,650	−66	22
C	Fundamental	43,517	$-3 \cdot 0_1$	70
D	Fundamental longitudinal	413,600	−27	60
	Fundamental in axis direction	385,260	−51	35
	Fundamental in axis direction after grinding twinned portion away	337,750	−80	47
E	A. Hilger complete resonators	123,031	Less than 0·2	—
F	,, ,, ,, ...	470,800	−16	45

TABLE VI.—Transverse Resonators, also Usable as Oscillators.

Bar.	Mode of vibration.	Frequency, cycles per second.	Temp. coeff., parts in a million.	$\phi_2 \times 10^6$.
G	Fundamental transverse...	847,050	−53	53
H	,, ,, ...	472,150	−34	22
I (disc)	Complete mounted crystal in disc form, fundamental transverse	455,500	−26	43
	Fundamental longitudinal	75,300	−40	47
	,, ,, in direction of optic axis	106,230	−74	40

These results are of great interest, but can only be considered fragmentary. Much valuable information would be gained by a complete investigation of the temperature coefficients with special reference to the proportions of length, width and thickness of the bar or plate and with respect to various air gaps. The ratios of the various frequencies could doubtless be made to yield information regarding the mode of vibration when considered in conjunction with the mathematical theory of resonance in bars or plates. These considerations require a complete investigation in themselves and lie outside the scope of the present paper.

Returning to the temperature coefficients of frequency, it is seen that for the fundamental mode, and for the overtone modes in three segments, the coefficients are very small. The coefficients in the direction of the electric axis (transverse mode of vibration) and in the direction of the optic axis (as in disc I) are very much larger. Measurements

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of the coefficients of linear expansion and of YOUNG's modulus have been made by various experimenters, and extracts from their results are quoted in a recent paper by BRAGG and GIBBS* on the crystal structure of quartz.

I am not at the moment aware that the coefficients of YOUNG's modulus in a transverse plane of the crystal, and specifically along and perpendicular to the electric axis, under conditions of pure tensile stress in these respective directions, have been made, but it would appear that the differences are very great, and that the coefficient of YOUNG's modulus in directions perpendicular to the electric and to the optic axes is smaller than has hitherto been supposed. The method of frequency change is, at any rate, a specially suitable one for the purpose of carrying out such measurements.

The larger coefficients for the higher overtone modes of vibration occur, I think, as a result of the shortness of the vibrating segment. With a segment of length of the same order as the width of the bar, the modulus along the crystal axis will exert an influence on the mode of vibration and on the frequency, so that the stresses are no longer even approximately pure, such as will be the case for the fundamental mode of longitudinal vibration of a thin rod, in which a transverse plane section will vibrate with a nearly pure translational motion.

In those cases in which it was observed, the value of ϕ_2 is included in the Tables for the various resonators in their various modes of vibration. It will be seen that this quantity varies irregularly with the mode of vibration in the same bar, due, no doubt, to the non-uniformity of the quartz. It is practically impossible to obtain a specimen bar 6 cms. long which is entirely free from visible twinning when examined under the polariscope. If a mixed portion of the bar comes at a node for a particular mode of vibration, it is certain to cause a diminution in amplitude and an increase in the log. dec. of the vibrations. It may be that the anomalous temperature effects in bar B are associated with the variations in ϕ_2.

Three conclusions which may be drawn from these measurements of temperature coefficient are :—

(1) That the temperature coefficient of response frequency for a thin bar when vibrating longitudinally is small and in a good specimen is of the order -5×10^{-6} per 1° C. rise in temperature.

(2) The temperature coefficient of response frequency of a disc or plate vibrating transversely in the direction of the electric axis is of the order -40×10^{-6} per 1° C. rise in temperature and is, therefore, by no means negligible. Values ranging from -30 to -70×10^{-6} have been observed.

(3) The temperature coefficient of response frequency of vibration in the direction of the optic axis is larger than either of the foregoing and may reach -80×10^{-6} per 1° C. rise. If, in the case of a disc or plate, the dimensions along the optic axis

* Sir WILLIAM BRAGG and R. E. GIBBS, 'Proc. Roy. Soc.,' A, Vol. 109, p. 405, 1925.

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are of the same order as those in the plane perpendicular to it, the temperature coefficients of all the modes of vibration are large.

A number of properties of resonators not embraced by the foregoing theory, but nevertheless of importance for design, were investigated and may be snmarised shortly in conclusion.

Effect on Response Frequency of Displacement of the Resonator from its Position of Centrality in a Fixed Air Gap.—Observations were made of the change in response frequency when a resonator was displaced transversely by successive amounts from the position of centrality within the gap to the position in which one face of the resonator was practically in contact with the electrode. The resonator experimented upon was a bar 62 mms. long, 7.5 mms. deep, and 3.3 mms. thick, the last-named dimensions being in the direction of the electric axis.

The observations were only made upon the fundamental longitudinal mode of vibration, which had a frequency of about 43.8 kilocycles per second.

The results for two different air gaps are given in Table VII.

TABLE VII.

Displacement of Bar, mm.	Frequency change, parts in 10^6.	
	Total gap 0·7 mm.	Total gap 2·7 mms.
0	0	0
0·1	− 8	− 2
0·2	−32	− 8
0·3	−80	− 18
0·5	—	− 44
0·7	—	− 78
1·0	—	−153
1·35	—	−284

According to theory, there should, of course, be no change in response frequency whenever the bar is within the gap, since the capacity K_2 consists of two condensers K_a, K_b, in series, where a and b are suffixes to identify these capacities with air gaps a and b on either side of the resonator.

$$K_2 = \frac{K_a K_b}{K_a + K_b}, \qquad \text{but } K_a = \frac{B}{A} \quad \text{and} \quad K_b = \frac{B}{b},$$

where B is a constant. Hence

$$K_2 = \frac{B_2/ab}{B/a + B/b} = B/(a + b) = \text{const.}$$

The displacement of the resonator does not alter $a + b$, and hence K_2 does not change. We have already seen, however, by the curve of frequency change (Fig. 16) that the actual change in response frequency is greater than that predicted by theory, and that

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this is probably due to a change in K consequent upon the reaction of the self-induced electric field upon the effective permittivity of the quartz. It is to be expected, therefore, that the asymmetry of this induced field resulting from the displacement of the resonator will have the effect observed.

Whether the above explanation is correct, or whether the cause of the effect is that the electric flux due to the applied field does not remain constant when the resonator is asymmetrical, requires further examination. It is, however, quite clear that the effect is by no means negligible from the standpoint of the use of such resonators as frequency standards.

The bar here quoted as an example is relatively thick; in practice the use of a thickness of 3 mms. on a bar 60 mms. long would be considered rather extravagant. With such a bar a possible variation of response frequency of one part in ten thousand can occur if it is mounted with a total gap of 0.7 mm. and no precautions are taken to locate it within the gap. In practice it would be more probable that a thickness of 1.5 mms. would be used for the bar. In such a case a similar frequency range would occur if the total gap were only 0.3 mm. With a smaller gap the total range of frequency due to displacement will be smaller, but on the other hand, the rate of change of frequency is greater. In the case of a bar 1.5 mms. wide with a total air gap of 0.3 mm. the rate of change of frequency with change of air gap would be about + 4 parts in a hundred thousand for an increase in air gap of 0.01 mm. A reduction of the air gap to 0.2 mm. would reduce the changes due to asymmetry, but would increase those due to a possible change of 0.01 mm. in the actual gap.

As in so many other cases the best design is a compromise between conflicting conditions, and depends chiefly upon the permissible cost in materials and construction.

Experiments on Oscillators.—Although this paper deals mainly with the analysis of piezo-electric crystals as resonators, because such analysis can be more easily undertaken on resonators than on oscillators, a considerable amount of experience has been gained with respect to the behaviour of crystals mounted and used in conjunction with a triode in such a manner as to be self-maintaining. The advantages of such a method of use need no elaboration; the use of crystals in such a manner has been described by CADY, PIERCE and other experimenters.

With regard to the effects of variation of the air gap upon the frequency of maintenance of oscillators, a number of experiments made by the author have shown that the changes are exactly parallel with those obtained on the same specimen when tested as a resonator. Similarly, also, with regard to temperature coefficient: this has been found to be the same whether the crystal is tested as an oscillator or as a resonator.

There seems no reason to doubt, therefore, that a similar equivalent electrical network will account for the behaviour of the crystal when functioning as an oscillator as when used as a resonator.

The response frequency as a resonator and the frequency of the self-maintained vibrations as an oscillator are, however, considerably different; in the latter case the

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frequency is higher so that the crystal, considered as a two-terminal piece of apparatus, is behaving as a large positive impedance at the self-maintaining frequency. A complete mathematical treatment of the mounted crystal when attached to a valve system is necessary before the significance of this fact can be appreciated. This is not attempted in the present paper, but the changes in frequency of an oscillator consequent upon the addition of a small capacity across the grid-filament circuit suggest that the internal valve capacities are an essential factor in the maintenance of the oscillations and that the grid filament effective input capacity is responsible for the observed frequency difference between resonator and oscillator.

The conditions of operation are, however, more severe for an oscillator. The quartz itself has to be chosen with particular care. In general, a better specimen must be chosen for an oscillator than for a resonator. It must be ground with more care and the faces must be left of a finer matt surface without noticeable scratches. The air gaps have of necessity to be much smaller in order that sufficient reaction on the valve may be obtained to enable it to supply the necessary energy for maintenance of the oscillation. In one simple arrangement the crystal is in the form of a disc or rectangular plate and rests by its own weight on the lower electrode, whilst the upper electrode is either lightly pressed upon the upper surface or is otherwise supported, leaving a very small air gap. Observations have shown that considerable variations of frequency are possible—of the order of one part in ten thousand—depending upon the exact condition of the crystal. There appears to be a settling-down or movement into closer apposition with the lower electrode, resulting in a slow drift in frequency with time and an increase in the log. dec. of the oscillator so that it is maintained less vigorously in vibration under fixed external conditions. The normal behaviour is usually restored upon turning the mounted crystal upside down and replacing it or by tapping it; it seems necessary that it should "float" on a certain minimum thickness of air in order to operate satisfactorily. I have found that the use of three minute supports for the crystal leaving an air gap of 0.1 mm. results in a satisfactory behaviour.

Considerable difficulties arise in the construction of oscillators for frequencies higher than 3×10^6 cycles per second. The thickness must be uniform to about 0.001 mm. and the proportions of width and length to thickness appear to be of considerable importance. Further investigation on the technical aspect of the production of satisfactory high-frequency oscillators is necessary.

Conclusions and Comments.

(1) The conclusions to be drawn from the experiments and simple theory given here would appear to be that an electrical network of the kind first suggested by Butterworth can explain in a general way the behaviour of a piezo-electric resonator or oscillator. It may be mentioned in passing that before finally using Butterworth's system the writer had developed the system given in Fig. 31 and had found it to give

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a satisfactory explanation of the resonator. It is, in fact, interpretable almost exactly algebraically into the system of Fig. 2. The system actually adopted is, however, somewhat more convenient to use and satisfies the imagination as corresponding more nearly to the physical truth.

With regard to the analysis of the resonator, in order to find K, K_1 and S, a matter of considerable theoretical interest which has purposely been omitted until now must be mentioned. An inspection of the equivalent electrical circuit of the resonator shows that at frequencies outside the region where $NK\omega^2$ is nearly unity, the resonator should revert to a simple series circuit consisting of K_1 and K_2 in series, K_2 being the air-gap capacity and K_1 the quartz capacity, calculable from its dimensions and known dielectric constant. K_1 can, of course, also be measured by a direct replacement method at frequencies on each side of the response frequency.

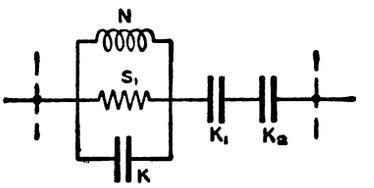

FIG. 31. Alternative Equivalent Network for a Resonator.

In the case of the longitudinal resonator the calculations and the measurements were made. The value found for K_1 was 11.0_7, $\mu\mu F$, whereas the value at resonance deduced from the crevasses was found to be 8.0_7, $\mu\mu F$. It seems clear, therefore, that the effective value of K_1 at resonance is definitely smaller than the geometrical value, as a result of the self-induced electric field. These measurements are difficult, and the probable error in the experimental determinations in the two cases is certainly not less than 0.3 $\mu\mu F$. It is believed, however, that this difference is real. It is interesting to recall in this connexion that in the note by VAN DYKE* referred to in the footnote on p. 149 the statement is definitely made that K_1 at resonance is given by an expression of the form

$$K_1 = \left(\frac{K}{4\pi} - \delta\varepsilon\right)\frac{bl}{e},$$

where $Kbl/4\pi e = K_1'$ is the geometric capacity and $\delta\epsilon \,.\, bl/e$ is a capacity resulting from the reaction of the electric field upon K.

It is interesting to find that the difference observed is in the direction predicted by the theory of VAN DYKE, but the amount found in the experiments described here is much greater than the 1 per cent. found for a resonator of CADY'S. It is probable that

* K. S. VAN DYKE, ' Phys. Rev.,' Vol. 25, Abs. 52, p. 895, 1925.

It has been worked out from first principles that the constants of the network (in terms of the nomenclature used in the present paper) are as follows :—

$$N = M/4\varepsilon^2 b^2, \qquad K = 4\varepsilon^2 b^2/g, \qquad S = N_1/4\varepsilon^2 b^2,$$

$$\text{and} \quad K_1 = \left(\frac{k}{4\pi} - \delta\varepsilon\right)\frac{bl}{e},$$

where

M, N_1 and g are the equivalent mass, frictional coefficient and stiffness of control,

δ is the modulus of elasticity, and

ε is the piezo-electric constant in C.G.S. electrostatic units.

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the term $\delta\varepsilon$, which is given for the case in which the electrodes are in contact with the quartz, will be multiplied by a considerable factor when there is an air gap, since under these latter conditions the voltage gradient in the quartz is enormously increased.

Thus we have seen, from equation (*a*), p. 151, and Fig. 4, that the resonator behaves at the response frequency almost exactly as a non-inductive resistance of value $S \cdot K_t^2/K_2^2$, by substitution of $-K/K_t$ for $\dot{q}$. In a case where $K_1 = K_2$ the effective resistance becomes 4 S.

The r.m.s. current I_3 taken by the resonator will be $E_2/4S$, where E_2 is the r.m.s. voltage applied to the terminals. But the voltage across K_2 will be given by $I_3/K_2\omega$; hence the ratio of these voltages is, in the case taken, equal to

$$\frac{1}{4SK_2\omega} = \frac{1}{\phi_2}\,\frac{K}{4K_2}.$$

Assuming normal values of 0.001 for $K/4K_2$, and 30×10^{-6} for ϕ_2, the ratio of voltages becomes 30. The average r.m.s. voltage across the air gaps is therefore about 30 times that applied to the electrodes. This fact has been used in a beautiful manner by GIEBE by placing the mounted resonator in a vacuum so that a visible glow occurs in the air space when the response occurs. The mean r.m.s. voltage across the quartz will therefore also be about 30 times the applied voltage. The gradient over the middle portion of the bar will be considerably greater than that towards the ends. The voltage gradient is a maximum at the centre of the bar, and has approximately a sine wave distribution from the centre to the ends.

In the case of the transverse resonator completely analysed, the similar measurements at frequencies on each side of the response frequency were made. In this case, however, the statical capacity found for K_1 was 1.6 $\mu\mu$F, whereas the vibrational capacity found was 1.9_3 $\mu\mu$F, so that the difference here is reversed in sign. CADY does not state whether the vibrational capacity K_1 should be greater or less than the statical K_1' for a resonator vibrating transversely in the form of a plate under the longitudinal piezo-electric effect. The measurements in the case of the small transverse resonator here investigated are, however, even more uncertain than those on the longitudinal resonator, owing to its shape. No great weight can therefore be attached to this difference.

With regard to the calculation of K_1' and K_2, also from the area of the quartz and from the air gap, it may be objected that it is not justifiable to do this using the simple formula on the assumption of an undisturbed electric field. It is realised that to a considerable extent this is true, but at the smaller air gaps the effect will be very small, and at the larger air gaps K_2 has a value which is small compared with K_1', and does not exert much influence on the behaviour. In any case, the straightness of the lines deduced from observations of $\sigma_{1\,\mathrm{min.}}$ with various air gaps and with various values of ϕ_1 is undoubted, so that the question of an error in K_2 then really resolves itself into a question

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of units, viz., that K, K_1 and K_2 are all in the same units, but that these may not be strictly $\mu\mu$F units, but that the units are such that, by calculating from the simple capacity formula, the behaviour of the resonator is consistently explained when used on an external circuit in which the unit of capacity is the $\mu\mu$ F.

Further work is needed on resonators of larger capacity in order to elucidate the departures of the behaviour of the resonator from the predictions of the theory, in particular in respect of K_1, which is definitely different from its value at frequencies outside the belt of response owing to the self-induced electric field. In this respect it

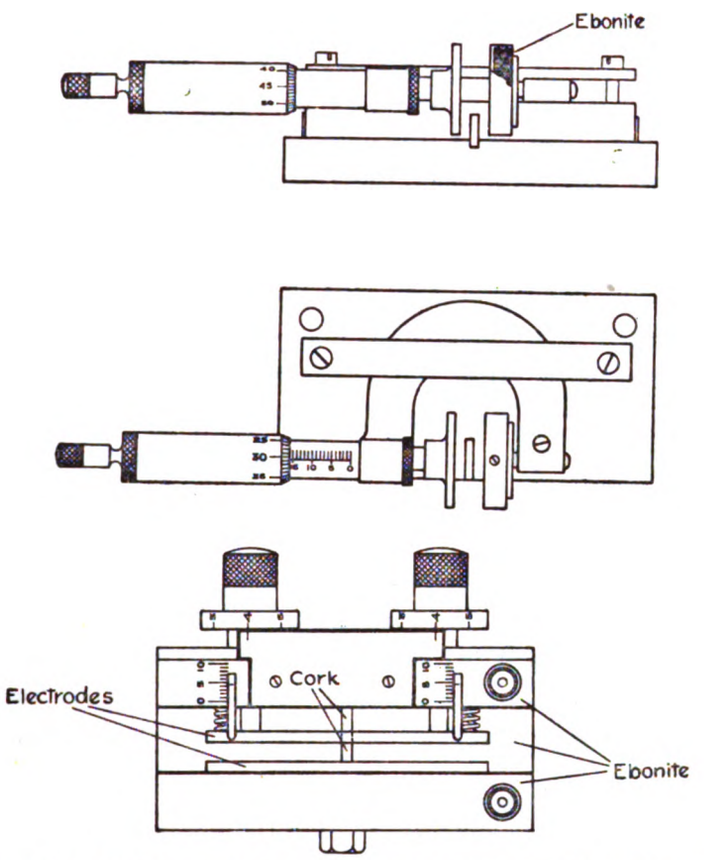

FIG. 32. Adjustable Air Gaps for Observations on Resonators with various Values of K_2

is probable that, as found at the point of maximum response, K_1 really increases from this value to nearly its geometric value as the value of σ_1 rises to unity on each side of the crevasse. This change cannot easily be elucidated from the curves of σ_1 or σ_2, since K_1 is of comparatively small influence on the shape of these curves.

The reaction of the self-induced electric field on K, acting as a cause of departure of the response frequency change from that given by theory, will also repay further investigation.

In connexion with such more complete investigations, the use of a separate small variable air condenser in series with the resonator has not been overlooked as an

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alternative to varying K_2. Considerable constructional and experimental difficulties arise, however. The construction of a screened variable condenser which can be varied from about 2 $\mu\mu$F to 20 $\mu\mu$F does not appear to have been solved. The high insulation resistance and small dielectric loss necessary, if useful determinations of ϕ_2 and $\sigma_{1\ \text{min.}}$ are to be made, will probably give great difficulty. This avenue would, however, certainly be a useful one to pursue, as it would be possible to work with a fixed known air gap, and so keep all the resonator constants invariable when the series capacity is varied.

(2) The temperature coefficients of response frequency are not negligible except for longitudinal resonators of suitable shape. The coefficients are those to be anticipated from the known coefficients of YOUNG's modulus with temperature. Further information regarding the latter could be obtained by a systematic investigation of the frequency-temperature coefficients of resonators of various types and modes of vibration. Care would be needed to discriminate a possible effect due to a change in K_1 with temperature. Observations of the coefficients with the same resonator in different air gaps would serve this end.

(3) The simple theory developed accounts for those effects on oscillators which are common to resonators; it is probable that a more complete theory introducing the electrical equivalents of the valve and other associated electrical circuits would account for the further effects on frequency of oscillators consequent upon these circuits. Such effects have been studied to some extent, and are being pursued. Scope for considerable investigation exists in the case of oscillators both from the theoretical and from the experimental points of view.

A considerable future awaits development in the application of resonators and oscillators as frequency standards, filters, frequency controllers, etc. This use at transmitting stations, conjointly with the production of local heterodyne frequencies by the stations, will permit of the production of an extremely constant frequency difference. This is an application that has considerable possibilities not yet developed.

In conclusion, I wish to render thanks to Mr. J. E. P. L. VIGOUREUX, M.Sc., and to Mr. W. D. MARTIN, who have shown much care and patience in taking many of the observations, some of which are tedious. The former has also shown much insight into the experiments, and the latter has prepared some of the specimens with skill.

The work was carried out under the auspices of the Standards Sub-Committee of the Radio Research Board, Department of Scientific and Industrial Research.

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IMPROVED CATHODE-RAY TUBE METHOD FOR THE HARMONIC COMPARISON OF FREQUENCIES

BY

D. W. DYE, D.Sc., A.C.G.I., F.R.S.,

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ABSTRACT.

Convenient arrangements for the superposition of a telephonic and a radio-frequency displacement of the light spot of a cathode-ray tube are described. The arrangements include the following :—

(1) Circular or elliptical trace at a telephonic frequency to serve as a time axis. When great steadiness and uniformity are desired, a valve-maintained tuning fork controls the rate of revolution of the light spot.

(2) Radial, circular and straight-line displacements at low radio or high audio frequencies are next described, and examples of the stationary patterns produced when these are superposed on the circular or elliptical time trace are given.

(3) A special application of the method to the delineation of wave forms consists in the superposition of a long narrow elliptical time displacement and a straight-line high-frequency displacement in a direction at right angles to the major axis of the ellipse. By this means a considerable portion of the time ellipse is nearly a straight line representing a uniform time axis. The wave shape at the high frequency then appears in normal form and may be made of quite considerable size.

Some examples of each of the foregoing methods are given in the form of reproductions from prints made directly on pieces of bromide paper held down on the top of the cathode-ray tube for a few seconds (3 to 10).

A special case occurs when an oscillatory valve system is arranged with grid and anode circuits closely coupled and harmonically resonant to each other. Wave forms consisting of a fundamental and a resonant single harmonic can be produced in this manner. Examples of current wave shapes in the grid and anode circuits are given, together with the voltage wave shape induced in a loosely-coupled untuned coil.

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IMPROVED CATHODE-RAY TUBE METHOD FOR THE HARMONIC COMPARISON OF FREQUENCIES.

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[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 37, 1925.]

The cathode-ray tube has been used by a number of investigators for the delineation of wave forms and the determination of harmonic ratios of radio frequencies. Various methods have been adopted to provide a convenient time base and to arrange so that confusion of the trace obtained is avoided. Of the methods adopted, mention may first be made of the use of Lissajous' figures* for the harmonic comparison of two radio frequencies. The chief limitation of this method is that ratios greater than about 15 : 1 introduce such complicated patterns that their interpretation and photography are difficult.

A method worked out in great detail by Dufour† consists in producing a zig-zag time trace across and down the screen. The high-frequency wave is superposed upon this in a direction at right angles to the time trace. For certain kinds of work this method is valuable, and allows results to be obtained which otherwise would be impossible. The equipment is very elaborate and expensive, and does not conveniently permit of a stationary pattern being formed. It is, therefore, necessary to take a photographic impression directly upon the plate inside the tube. This latter necessity is, of course, inseparable from any method in which photography of a transient or non-recurring phenomenon is required. In many cases, however, the oscillations under measurement or investigation are steadily maintained. It then becomes possible to produce a stationary pattern by arranging so that the path followed by the ray is repeated after a short time interval.

More recently Kipping‡ has used a neon tube trigger arrangement for producing a time trace which possesses some advantages and is very simple. In Kipping's paper the idea of using the rotating ray method about to be described is suggested amongst others, but the idea originated a considerable time before with the present writer, and was successfully used at the National Physical Laboratory in 1923. As long ago as

* F. Braun, 'Ann. der Phys. u. Chem.,' Vol. 60, p. 552, 1897; 'Elektrotech. Zeits.,' Vol. 19, p. 204, 1898; Simon and Reich, 'Phys. Zeits.,' Vol. 2, p. 284, 1901; Hazen and Kenyon, 'Bureau of Standards Sci. Papers,' Vol. 19, Paper 489, p. 415, 1924.

† A. Dufour, 'J. de Physique,' Vol. 1, p. 147, 1920.

‡ M. V. Kipping, 'Wireless World,' Vol. 13, p. 705, 1924.

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1899 ZENNECK used a rotating ray for low-frequency measurement. E. LÜBKE* has also used a rotating ray for the purpose of delineating wave shapes at high frequencies; in this method the exact analogy of the point-by-point method is used. The circularly rotating ray acts as a switching device by first passing through a radial slit, and thence between a pair of plates. The instantaneous conduction thus provided between the plates allows a voltmeter or electrometer to become charged to the corresponding instantaneous voltage. By slowly rotating, a few degrees at a time, the system producing the circularly rotating ray, the unknown wave-shape may be plotted out. The method, of course, requires special attachments within the tube to enable the system to work. The tube was of the heated cathode type, and operated with 400 volts anode potential.

The rotating ray method—which does not seem to have received much attention—can, however, be used with great advantages in many cases. In particular the method is valuable for the purpose of setting a low radio frequency to an exact value for purposes of calibration, etc. The method consists simply in producing a circular or elliptical time trace by means of two-phase voltage of constant and low frequency. The radio frequency may then be caused to operate on the ray in one of three different ways, as follows :—

(1) A small voltage (about 20-30 volts) at the radio frequency is introduced into the steady voltage supply to the cathode-ray tube. By this means the ray receives a radial vibratory displacement due to the radio frequency, at whatever position it may be in its circular or elliptical path at the audio frequency. This is due, of course, to the variation in velocity of the electrons forming the ray consequent upon the variation of anode voltage.

(2) A small circular movement may be imparted to the ray at the radio frequency whilst the large circular or elliptical movement is provided by the audio frequency.

(3) A vibratory movement, in a fixed direction, may be given by the radio frequency, whilst the large circular or elliptical movement takes place at the audio frequency.

The choice of these three methods is determined by the particular object in view in any given case.

Convenient circuit arrangements for carrying out these three methods have been worked out to suit the Western Electric tube with heated cathode, and are described below after a short account of the best means of producing the rotating ray.

The most accurate method of producing the circular time base is to use a valve-maintained tuning fork as a fundamental time standard. A small voltage obtained from the fork is amplified up by a two-stage amplifier, so as to provide a source of voltage of very constant frequency. This voltage is converted to a two-phase source by the help of a

* E. LÜBKE, 'Arch. Elektrotech.,' Vol. 6, p. 161, 1917; for translation see 'Electrician,' Vol. 83, p. 270, 1919.

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condenser and a virtual non-inductive resistance. These two voltages of adjustable amplitude and phase relationship are applied to the two deflecting plate systems of the cathode-ray tube.

A convenient set-up for the method is shown in Fig. 1 in diagrammatic form. The tuning fork is shown at A, and is of ordinary form. The small voltage given by the output winding will not in general be of good wave form. In order to improve this the anode circuit of the second valve of the amplifier is a tuned circuit, consisting of the air core self-inductance L_1 and adjustable condenser C_1. The coil L_1 is best made with

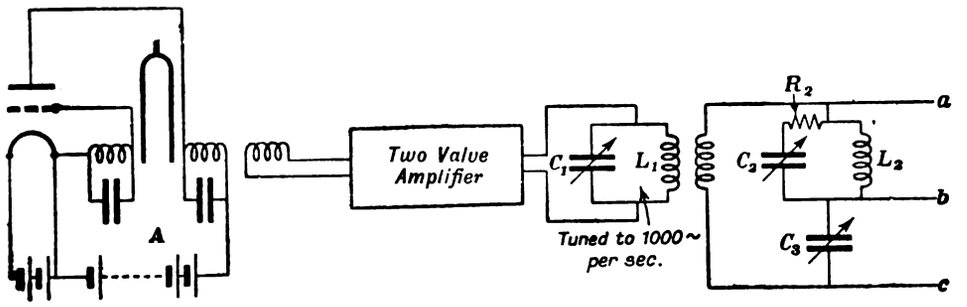

FIG. 1. Arrangement for Producing Circular Time Trace.

a generous amount of wire, so as to be of fairly good time constant, with inductance of about 0.5 henry. A secondary winding closely coupled to L_1 provides the supply of nearly sine-wave form. This is connected to the phase-splitting arrangement whereby the two quadrature components are produced.

The phase-splitting arrangement consists of the condenser C_3 in series with a tunable resonant circuit L_2, C_2. This circuit, when adjusted to resonance, behaves as a pure non-inductive resistance of value adjusted by the resistance R_2. It has been found desirable to use a virtual resistance of this form in preference to an actual resistance. The reason for this is that when an actual resistance of, say, 10,000 ohms is used, this resistance has also a high impedance at the radio frequency. It then becomes almost impossible to keep a small parasitic radio-frequency voltage from finding its way on to the deflecting plates connected to this resistance. By the device indicated, a truly non-inductive resistance is provided at the audio frequency, whilst at any other frequency not very near the resonant one the impedance is very low.

The two quadrature voltages are applied to the deflecting plates, and by adjustment of C_2 and R_2 a nearly perfect circle is obtained after tuning C_1 to resonance.

The circle is adjusted to any desired size by adjustment of the brightness of the filament of the second valve of the amplifier. It can, of course, be made an ellipse of any eccentricity by slight detuning of L_1, C_2, or by adjustment of R_2 or C_3. In cases where it is not important to know the radio frequency exactly, the tuning fork may, of course, be dispensed with, and a low-frequency oscillator used instead. The freedom for the choice of a suitable rotation frequency is then a great advantage in many cases.

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The valve oscillator and coupling coils are as shown in Fig. 2. This, at first sight, appears to be of normal type, but the arrangement of the capacities and windings L_1, L_2 permits of the production of resonant harmonic wave forms as discussed later on.

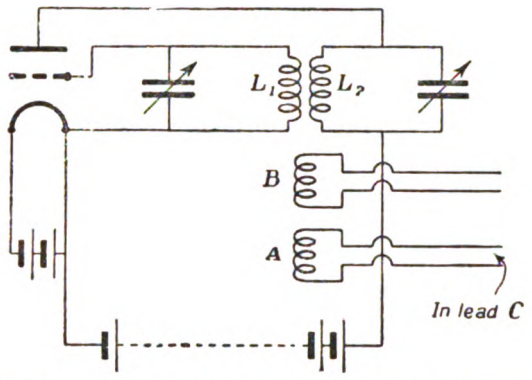

FIG. 2. Valve Oscillator with Resonant Harmonics.

The coils A, B are for the purpose of applying the radio-frequency voltage to the cathode-ray tube in a number of different ways to suit different requirements.

When required, coil B is connected to a pair of coils wound on a double ebonite bobbin of such form and proportions that they produce a uniform magnetic field which is transverse to the axis of the tube. These coils are shown in diagrammatic form in Fig. 3, at F_1, F_2, which indicates the general cathode tube arrangement. The bobbin

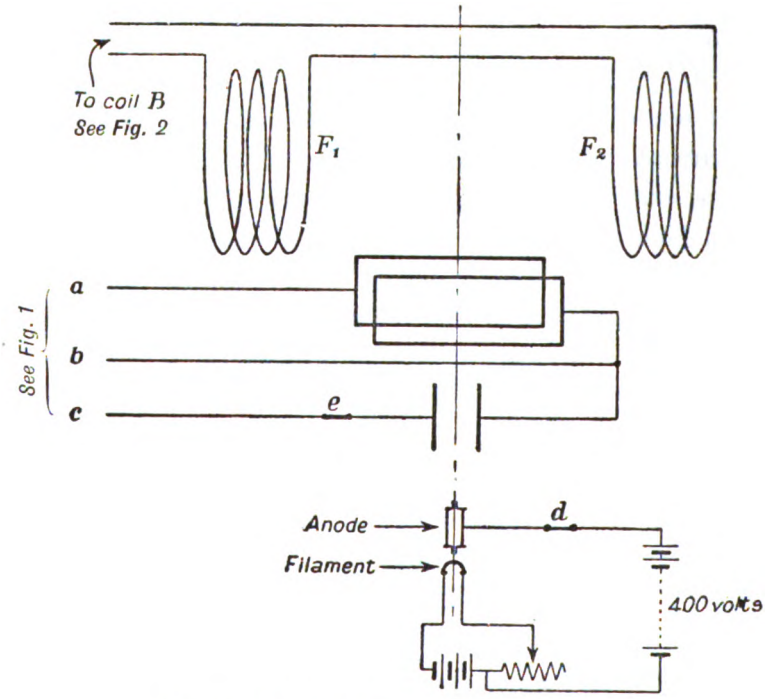

FIG. 3. Cathode Ray Tube Arrangement.

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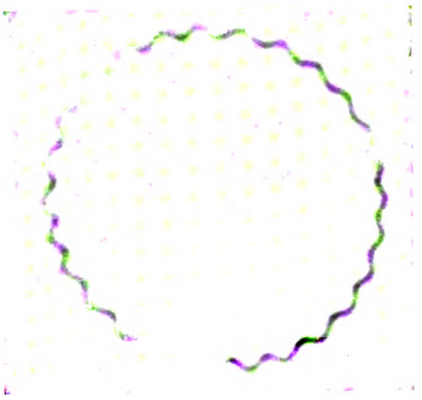
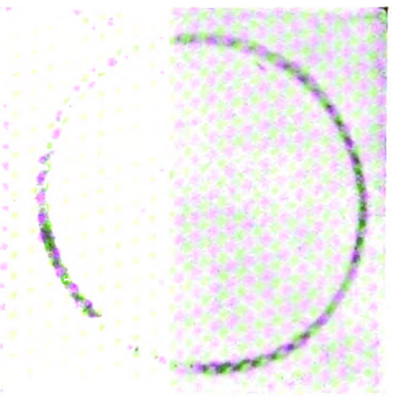


(*a*) Radial high-frequency deflection of 26,000 cycles per second on a circular time base at 1,000 cycles per second.



(*b*) Radial high-frequency deflection at 26,500 cycles per second on a circular time base at 1,000 cycles per second.

FIG. 4.

[*To face p.* 210.

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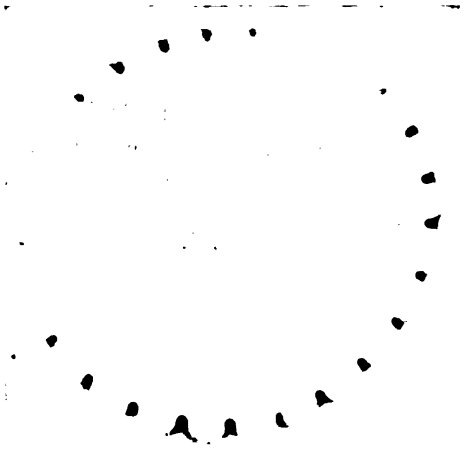
(*a*) Circular high-frequency at 26,000 cycles per second rotating in same direction as the circular time base of 1,000 cycles per second.

(*b*) As (*a*), but the two circular motions are in opposite directions.



(*c*) Conditions as at (*a*), but the amplitude of the high-frequency motion has been adjusted to give a cusp.

FIG. 5.

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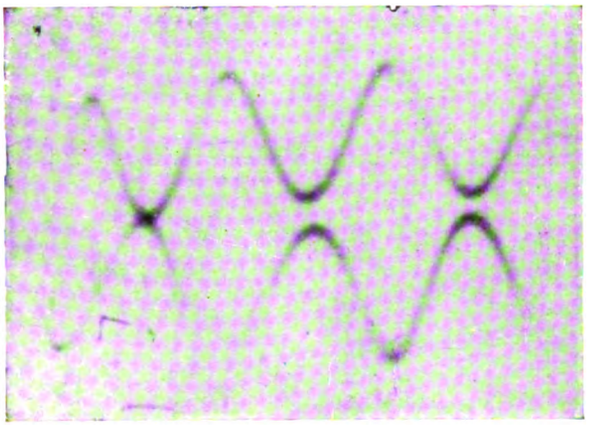
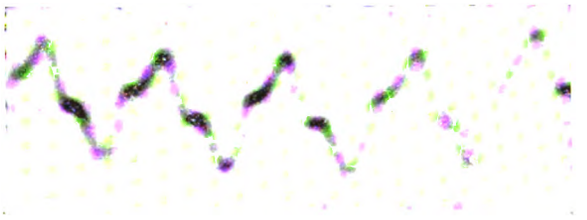


(*a*) Sine wave at a frequency of about 15,000 cycles per second. Long elliptical time base.



(*b* Same frequency as (*a*), but with a considerable proportion of a third harmonic.

FIG. 6.

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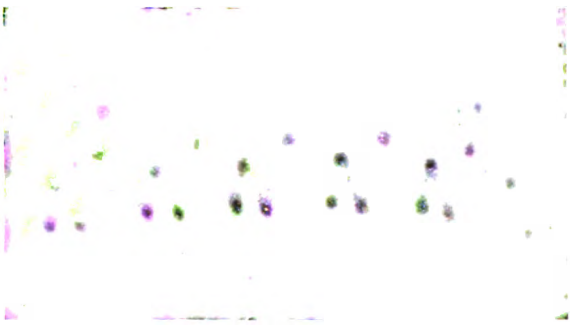
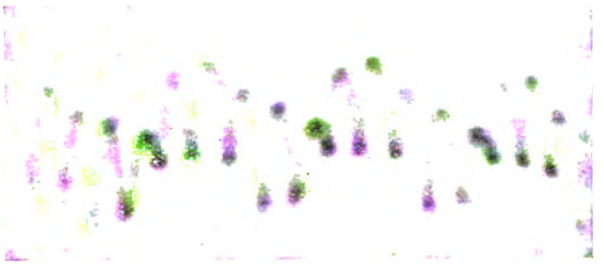


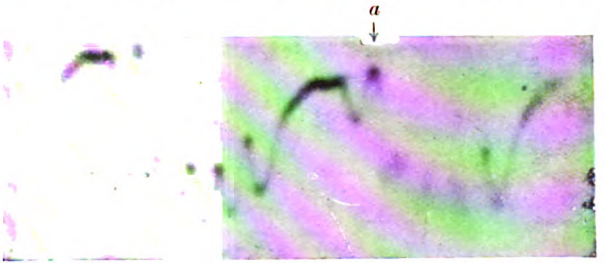
(*a*) Fundamental of 24,000 cycles per second with strong third harmonic.



(*b*) Similar to (*a*), but with a resonant fifth harmonic.

(*c*) Fundamental of 25,000 cycles per second with resonant harmonic giving 175,000 cycles per second.

(*d*) Showing interference between two neighbouring odd and even harmonics.



(*e*) A case similar to (*d*) for the seventh and eighth harmonics of a fundamental of about 12,000 cycles per second.

FIG. 7.

[*To face p.* 211.

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carrying these coils is made a sliding fit on the parallel part of the tube, and can be rotated so that the direction of the magnetic field can be altered in azimuth.

Coil A provides a radio-frequency voltage which may be introduced either into the anode battery circuit of the cathode ray tube at *d* in Fig. 3, or in the circuit of one pair of deflecting plates at *e*.

(1) An example of the pattern produced by the superposition on the time circle of a frequency of 26,000 cycles per second introduced as a voltage in the anode circuit is shown in Fig. 4 (*a*). This figure is reproduced from a photograph made by holding a piece of bromide paper down on the top of the cathode tube for about 3 seconds. The adjustment of frequency must, of course, be very precise in order that only a negligible drift occurs during 3 seconds. If, for example, a drift of one-tenth of a cycle is permissible during the 3 seconds, the constancy and steadiness of the high-frequency source must be equal to about two parts in a million. This can be secured by allowing everything time to settle down, and then catching the picture just as it passes through a stationary condition corresponding to harmonic synchronism.

The value of frequency is given directly by counting round the number of waves in the circle and multiplying by 1,000. The method not only allows exact frequencies such as 26,000 cycles per second to be obtained, but in addition frequencies such as 26,500 or 26,333 cycles can be set with perfect certainty. There is considerable difficulty in photographing these fractionally harmonic synchronous frequencies, since the light spot has to make two or three revolutions before retracing its path. A correspondingly longer exposure is necessary, and at the same time the permissible drift is reduced. A reproduction of such a case is shown in Fig. 4 (*b*). There is, however, no difficulty in setting a frequency to such values; in this manner a considerable number of standard reference frequencies are available for calibrating purposes.

One of the disadvantages of this method of superposition is that a rather high voltage at the radio frequency is necessary to produce the requisite radial deflection; a loss of definition occurs when the voltage reaches its minimum value. The radial movement permissible is also very limited.

(2) *Superposition of Circular Motions.*—The arrangement for this method can be seen from Fig. 2 and Fig. 3. The high-frequency rotation of the light spot is produced by the help of the field coils F_1, F_2, previously described. These produce a magnetic deflection in a plane which can be rotated by turning the double bobbin about the axis of the tube. The quadrature deflection is produced by introducing the high-frequency voltage given by inducing coil A into the circuit at the point *e* of Fig. 3. This voltage can also be obtained from the ends of the deflecting coils F_1, F_2, by suitably choosing the inductance of these. The phase adjustment is made by turning F_1, F_2, about the axis of the tube. The amplitude adjustment is made by sliding the coils F_1, F_2, up and down on the tube, or by variation of coupling of the inducing coils A and B. The small high-frequency circle is obtained and adjusted with the large low-frequency circle cut off.

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When both circular motions are allowed to operate simultaneously the beautiful circular looped patterns shown in Fig. 5 (*a*) and (*b*) are obtained. Two cases arise; the two circular motions may be in the same or in opposite directions. The corresponding patterns are shown in Fig. 5 (*a*) and (*b*), reproduced from photographic prints taken in the same manner as Fig. 4.

An inspection of the two pictures given in Fig. 5 shows that for the same high frequency the number of loops is two greater in one case than in the other. When the two circular motions are in the same direction, the loops are on the inside and the number of them is one less than the ratio of the frequencies. When the two circular motions are in opposite directions, the loops are on the outside and the number of them is one more than the ratio of the frequencies. By adjustment of the relative diameters of the two circular motions it is possible to obtain a sharp loop or cusp resulting from the instantaneous arrestment of the spot. This condition is very satisfactory photographically, and also renders the setting of fractional harmonics more certain. One of these latter is shown in Fig. 5 (*c*) for a case corresponding to (*a*). Although it is well-nigh impossible to photograph them, it is not difficult to set frequencies corresponding to seven or eight journeys of the light spot before the cycle is repeated. For example, a frequency such as 24,875 can be set by this means. The advantages of this method over the foregoing method are the much smaller high-frequency voltage necessary, and the formation of loops, which renders counting or photography easier.

(3) The arrangement is, however, not convenient for the examination of wave forms. For this purpose, therefore, the following method (3) is recommended. This method is merely a particular case of (2) just described. One of the components of the high frequency is suppressed—by removal of the deflecting coils F_1, F_2, for example. The *h.f.* circular motion becomes a straight line of fixed direction. The *l.f.* circular motion is then made elliptical with the major axis perpendicular to the *h.f.* direction. By making the eccentricity considerable we can greatly open out the time scale, and also render it nearly a straight line for a considerable length. The time scale will, of course, not be uniform right round the ellipse, but if it is long and narrow there will be a portion in the middle of the nearly straight part of sufficient length and uniformity of velocity to be quite satisfactory for delineating wave shapes at moderately low radio frequencies without distortion. The ordinate motion at the radio frequency may be made large when this method is used, and a reasonable size of wave obtained.

Examples of a good wave and of a wave with harmonics are given in Fig. 6. The nature of the waves with harmonics and the special uses of such waves will follow later.

The applications of the foregoing methods to telephonic frequencies are obvious; it is merely necessary to use a slower time base corresponding to a frequency of revolution of, say, 100 cycles per second, instead of 1,000 cycles. For quantitative work it is very desirable that the two motions should be proportional to their respective quantities; they should also be rectangular. For these reasons the high-frequency voltage should be introduced into the lead marked C on Fig. 3 at the point *e*, and the elongation of the

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ellipse must then be obtained by reducing C_3 to about a quarter of its original value. The rectangularity of the high-frequency motion with regard to the major axis of the ellipse can be seen by the eye when the two motions are taking place together, but at an inharmonic ratio. Adjustment is made by the condenser C_2.

The limits of frequency which may be standardised are determined largely by the tube itself. Owing to the relatively low velocity of the ray, the time occupied by the electrons during the passage between the deflecting plates is such that loss of definition occurs at frequencies above 200,000 per second.

This value cannot, however, be directly compared against a fundamental time base of one-thousandth of a second owing to uncertainty of counting. This trouble can be entirely overcome by the use of resonant harmonics, a brief account of which is given in the following section.

Resonant Harmonics in Valve Oscillators.—The distorted wave shown in Fig. 6 was known to be of a kind which commonly occurs in small valve oscillators such as heterodyne wave-meters and sources. It is, of course, well known that curvature of the valve characteristics and changes with amplitude of the internal anode and grid impedances can, and indeed must, introduce harmonics into the wave shape produced, but in the case shown and in many other cases the distortion was so large that it was felt that other special causes were responsible for the distortion.

The matter was therefore followed up somewhat. Observations of a power-valve oscillator at the Laboratory showed that when working in a normal manner the wave form was very good. The generator was such that normally the oscillatory circuit was of very small decrement and was in the anode circuit of the valve. The grid winding was of thin wire and had such mutual inductance to the oscillatory coil that the voltage on it was about one-fifth that on the oscillatory coil. The coupling between the coils was close. There was in this case no condenser across the grid winding. When, however, the main oscillatory circuit was in the grid circuit of the valve with the fine-fire coil as inductance, whilst the heavy-wire coil was in the anode circuit, the distortion was very pronounced.

A small variable condenser was accordingly placed across the anode winding. The value of this condenser was slowly increased whilst at the same time the frequency was kept constant by a careful simultaneous adjustment of the grid oscillatory circuit condenser. A remarkable series of changes occurred in the wave shape thus held approximately steady on the oscillograph. Harmonics such as the ninth, eighth, seventh, sixth, fifth, etc., successively became reinforced and reached large values as the anode condenser was varied.

When a harmonic is at its maximum amplitude the wave shape appears to consist almost entirely of the fundamental and this harmonic; such cases are shown in Fig. 7, (*a*), (*b*), (*c*) and (*d*). At intermediate settings of the anode condenser the wave shape passes through curious forms. For example, Fig. 7 (*d*) shows a shape corresponding to about equal proportions of two adjacent harmonics and very little else.

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These two harmonics reinforce each other during one half of the wave and cancel each other during the other half. In some cases the harmonic oscillation does not build up and die down smoothly during each half-wave, but commences as shown by the curious peak marked "*a*" in Fig. 7 (*e*), which refers to a similar case in which the seventh and eighth harmonics interfere.

These harmonic resonances correspond undoubtedly to a setting of the condenser of the anode oscillatory circuit such that the resonant frequency of this circuit bears an integral ratio to the main grid oscillatory circuit frequency. It is to be expected that the valve is able to produce the two corresponding frequencies simultaneously. It might be thought that two inharmonic frequencies could also be produced in the same manner, but such does not appear to be the case. The transition from one wave shape to another is smooth and continuous. It is not possible to obtain a stationary pattern and get a blurred figure at the same time, such as would occur if the two frequencies were inharmonic.

The wave shapes shown in Fig. 7 represent electromotive force induced in a coil coupled to the oscillator by mutual inductance. The voltage induced in such a case is, of course, proportional to ω, so that the harmonic components in the wave of induced voltage will be magnified in proportion to the order of the harmonic by comparison with the corresponding components of the current wave in the oscillator. The current wave in a practical case will therefore in general be much smoother than the waves of voltage shown in Fig. 7. In many cases, however, we are directly concerned with the voltage induced in a circuit. In such cases the wave shapes shown are those which must be considered.

In the actual oscillator used in these experiments we have two closely-coupled circuits, both carrying currents of different wave shapes and both coupled to the external coil in which the wave shape of voltage is produced. The voltage wave will therefore depend upon the two current wave shapes and the values of the two mutual inductances to the external coil. The phase relationship of the two current waves will also largely affect the induced voltage wave shape.

In Fig. 8 are shown three wave shapes corresponding to one condition of the oscillator and its circuits. The curves (*a*) and (*b*) are current wave shapes in the grid and anode coils respectively. Curve (*c*) is the voltage wave induced in a coil. The current waves were obtained by inserting the deflecting coils F_1 and F_2 directly into the circuit concerned. The coils were then oriented until the deflection was at right angles to the major axis of the ellipse. Unfortunately it was not possible to observe the phase relationship of the two currents. This could, however, have been done by the aid of two pairs of coils with axes at right angles to one another. The time circle would not then be used. A small Lissajous figure would be obtained in which a distorted ellipse would be formed by the two fundamental components each with its harmonic.

Referring to Fig. 8 again, the current waves produced are those which might have been expected. The anode circuit, of harmonic resonant frequency and small damping

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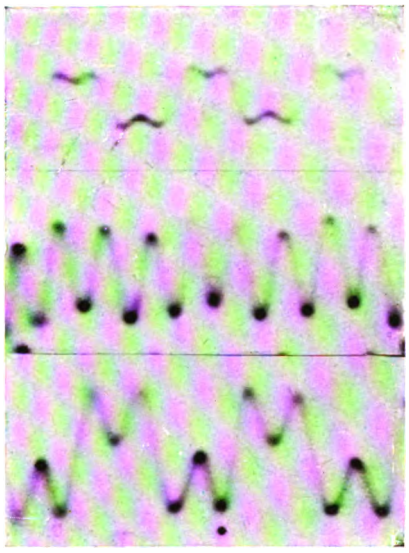

(*a*) Current in grid coil circuit of oscillator. Fundamental of about 20,000 cycles per second.

(*b*) Current in anode coil under exactly the same conditions as (*a*).

(*c*) Voltage induced in a coil loosely coupled to the oscillator.

FIG. 8.

[*To face p.* 215.

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coefficient, has a current wave shape of which the main component is the harmonic, but in which there is a noticeable amount of fundamental present. The grid circuit of resonant frequency of the fundamental has a greater damping coefficient than the anode circuit. As a consequence the wave shape has a considerable proportion of the harmonic component present.

The voltage wave shape is similar to those shown in the previous examples given.

The writer is not aware that this particular property of a valve oscillator has been observed before. For certain purposes such an oscillation is of great value. For example, a cascade system of oscillators all in harmonic synchronism can be built up whereby a very high radio frequency may be controlled from a lower one. For standardisation purposes the method is particularly suitable. One can, for example, use a fundamental frequency of, say, 30,000 cycles per second with a resonant ninth harmonic and so obtain a frequency of 270,000 cycles per second in a manner which leaves absolutely no uncertainty regarding frequency.

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XI.

THE MEASUREMENT OF THE INDUCTANCES OF FOUR-TERMINAL RESISTANCE STANDARDS

BY

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ABSTRACT.

The measurement of the effective inductances of " four-terminal " resistances or " shunts " is discussed. The method of measurement developed in the paper is an application of the Kelvin double bridge, used with alternating current, the phase-angle adjustment being obtained by condensers shunting the ratio arms. The procedure to be adopted for eliminating or evaluating corrections due to " leads " is described, and the advantages of the bridge in being free from stray fields, practically independent of frequency, capable of use with almost any desired current strength, and, moreover, very easy to work, are pointed out. Results of measurements made by this method on a series of typical shunts are recorded.

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THE MEASUREMENT OF THE INDUCTANCES OF FOUR-TERMINAL RESISTANCE STANDARDS.

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[Reprinted from the 'Proceedings of the Physical Society of London,' Vol. 39, 1927.]

Introductory.

Four-terminal resistance standards or "shunts" are widely used in precision alternating current measurements, their function being to provide a potential difference which is in phase with, and proportional to, the current flowing through them. Fig. 1

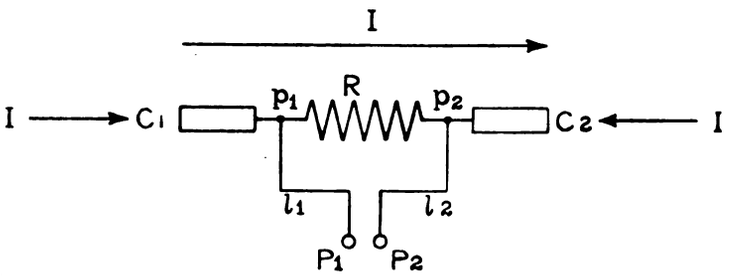

Fig. 1.

is a diagrammatic representation of such a shunt. The current I enters and leaves by the "current leads" C_1 and C_2. The potential difference measured is that existing between the "potential terminals" P_1 and P_2. In the ideal case, this potential difference is given by

$$V = RI,$$

where R is the resistance of the shunt. In actual practice, the potential difference V is never exactly in phase with the current I. We may regard it as consisting of two components, a large one, RI, in phase with the current, and a small one, jwLI, in quadrature with the current. Thus we may write

$$V = (R + jLw)\,I,$$

where j is the operator rotating through a right angle, and the frequency is $w/2\pi$. The quantity L in the equation is defined as the inductance of the shunt, while R is the resistance. The object of the present paper is to describe a method for the measurement of the inductance L.

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Two factors contribute to the quantity L, viz., the self-inductance of the shunt between the tapping points p_1 and p_2, and the mutual inductance between the potential leads l_1, l_2, and the whole circuit in which the current I flows. This mutual inductance component may oppose the self-inductance. In order that the inductance L may have a definite value for any particular shunt, it is obviously necessary that the current and potential circuits be arranged so that there can be no mutual inductance between the connecting leads which are attached to C_1 and C_2 and those attached to P_1 and P_2. Thus, the two current terminals must be near together, as also must the two potential terminals, and moreover these two pairs of terminals must not be near each other. In what follows it will be assumed that these conditions are satisfied. When using the shunts, the external leads connected to the potential terminals must be twisted together, and the leads to the current terminals must be similarly treated, since any open loop in the leads is likely to affect the mutual inductance between the potential and current circuits, and so have the effect of altering the value of L.

Previous Methods.

A number of methods of measurement of the self-inductances of standard shunts have already been described. A. CAMPBELL has given a general account of these in GLAZEBROOK'S "Dictionary of Applied Physics."* At the National Physical Laboratory, two alternating-current potentiometer methods, developed by Mr. CAMPBELL,† have been in use for some years. These methods make use of a mutual inductance as a standard of phase angle, and as it is generally necessary to work at telephonic frequency, errors due to impurity of the mutual inductance are liable to occur. The measurement is in any case a difficult one, since the circuits contain a number of inductive coils, and it is essential to make sure that the stray fields of these coils are not introducing an undesirable E.M.F. into any part of the circuit. Induction directly into the telephone detector is liable to be very troublesome. It was on this account that the author was led to consider the application of the Kelvin double bridge to this type of measurement. Arrangements of this bridge for alternating current measurements have been described by SHARP and CRAWFORD‡ and by WENNER,§ using inductances in series with the ratio arms for the inductance balance. SHARP and CRAWFORD remark that stray field effects are apt to be very serious when this is done, and on that account preferred to use a separately excited dynamometer as detector, obtaining the resistance balance by in-phase excitation, and adjusting for zero deflection, and then deducing the inductance relations by observation of the deflection with quadrature excitation. The procedure adopted by WENNER, though capable of high accuracy, appears to be somewhat exacting,

* GLAZEBROOK, 'Dictionary of Applied Physics,' Vol. 2, p. 441, Macmillan, London, 1922.

† A. CAMPBELL, 'N.P.L. Collected Researches,' Vol. 15, Paper 11, p. 165, or 'Proc. Phys. Soc. Lond.,' Vol. 29, p. 345, 1917.

‡ SHARP and CRAWFORD, 'Trans. Amer. Inst. Elec. Eng.,' Vol. 29, p. 1540, 1910.

§ F. WENNER, 'Bull. Bureau of Standards,' Vol. 8, Bull. 181, p. 559, 1912.

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since no less than three simultaneous double balances are required, and the stray field difficulty has to be attended to. In the method developed by the author, no inductive coils are used, and thus stray fields are reduced to a minimum. The quadrature balance is obtained by variable condensers shunting the ratio arms. It subsequently appeared that this same condenser adjustment had been used some time ago by SCHERING,* but as the development of the method is very different in the two cases, and the method used at the Laboratory is particularly easy to work, and need require no very special apparatus, it is considered worth while to describe it in some detail. Other methods of measuring the inductances of shunts are described by WENNER, WEIBEL and SILSBEE.† These, like CAMPBELL'S methods, are independent of standards of calculated inductance, but require special apparatus, and are not likely to be very quick in operation.

Theory of the Method.

Fig. 2 is a diagrammatic representation of the bridge. P and R are four-terminal shunts to be compared for inductance. Q and S are the main ratio arms, and α and β

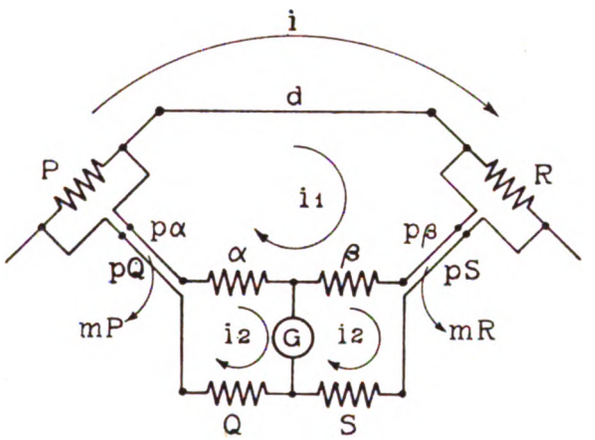

Fig. 2.

the auxiliary ones. Let P, R, Q, S, α, β be the impedance operators of the various coils. The potential terminals of the coil P are connected to the ratio coils αQ by means of twinned leads, and there is a similar pair of leads connecting the coils β and S to the potential points of R. Let p_α, p_β, p_Q, p_S be the impedance operators of these leads, and m_P, m_R their mutual operators; also let d be the impedance operator of the link connecting the current terminals of P and R. Let the mesh currents be i, i_1, i_2, as in Fig. 2. When the bridge is balanced the currents in the meshes αGQP, βRSG must be equal, as indicated (i_2).

* SCHERING, 'Elektrotech. Zeits.,' Vol. 34, p. 421, 1917.

† WENNER, WEIBEL and SILSBEE, 'Bull. Bureau of Standards,' Vol. 12, Bull. 246, p. 11, 1915.

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Applying KIRCHOFF'S equations to the network, we obtain as conditions of balance the equations

$$(i + i_2)\,\mathrm{P} + (i_2 - i_1)\,(\alpha + p_\alpha + m_\mathrm{P}) + i_2\,(\mathrm{Q} + p_\mathrm{Q} + m_\mathrm{P}) = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(i + i_2)\,\mathrm{R} + (i_2 - i_1)\,(\beta + p_\beta + m_\mathrm{R}) + i_2\,(\mathrm{S} + p_\mathrm{S} + m_\mathrm{R}) = 0, \quad . \quad . \quad . \quad . \quad (2)$$

$$(i + i_1)\,d - (i_2 - i_1)\,(\alpha + p_\alpha + \beta + p_\beta) - i_2\,(m_\mathrm{P} + m_\mathrm{R}) = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Rewriting (1) and (2) we have

$$(i + i_1)\,\mathrm{P} + (i_2 - i_1)\,(\mathrm{P} + \alpha + p_\alpha + m_\mathrm{P}) + i_2\,(\mathrm{Q} + p_\mathrm{Q} + m_\mathrm{P}) = 0, \quad . \quad . \quad (4)$$

$$(i + i_1)\,\mathrm{R} + (i_2 - i_1)\,(\mathrm{R} + \beta + p_\beta + m_\mathrm{R}) + i_2\,(\mathrm{S} + p_\mathrm{S} + m_\mathrm{R}) = 0. \quad . \quad . \quad (5)$$

Now let the connector d be removed, and the bridge again balanced. The mesh currents are now i_3 and i_4 (Fig. 3), and the conditions of balance are

$$(i_3 + i_4)\,(\mathrm{P} + \alpha + p_\alpha + m_\mathrm{P}) + i_4\,(\mathrm{Q} + p_\mathrm{Q} + m_\mathrm{P}) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$(i_3 + i_4)\,(\mathrm{R} + \beta + p_\beta + m_\mathrm{R}) + i_4\,(\mathrm{S} + p_\mathrm{S} + m_\mathrm{R}) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whence

$$(\mathrm{P} + \alpha + p_\alpha + m_\mathrm{P})/(\mathrm{R} + \beta + p_\beta + m_\mathrm{R}) = (\mathrm{Q} + p_\mathrm{Q} + m_\mathrm{P})/(\mathrm{S} + p_\mathrm{S} + m_\mathrm{R}) = k \text{ (say)}. \quad (8)$$

If now the bridge arms can be adjusted so that the bridge is balanced both when the

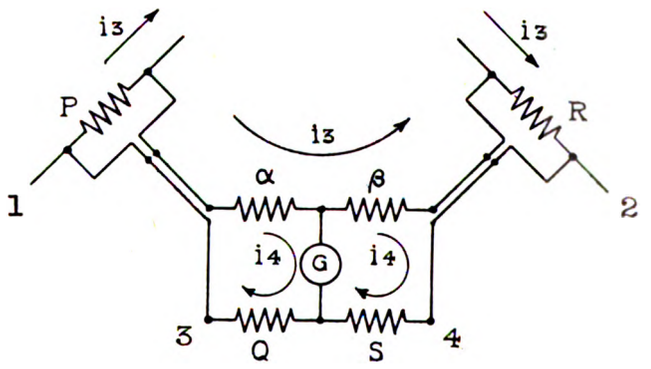

Fig. 3.

link d is inserted, and when it is removed, then equations (3), (4), (5) and (8) must hold for all values of the current i. Now, evaluating (4) $- k \times$ (5), we find, on substituting (8),

$$(i + i_1)\,(\mathrm{P} - k\mathrm{R}) = 0.$$

Thus, the condition for simultaneous balance is

$$\mathrm{P}/\mathrm{R} = k = (\mathrm{Q} + p_\mathrm{Q} + m_\mathrm{P})/(\mathrm{S} + p_\mathrm{S} + m_\mathrm{R}) = \mathrm{Q}'/\mathrm{S}', \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

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P, R, Q′, S′ are vector impedances. Let their phase angles be θ_P, θ_R, θ_Q', θ_S'. Then in order that equation (9) may hold, the relation between the phase angles must be

$$\theta_P - \theta_R = \theta_Q' - \theta_S'. \quad \ldots\ldots\ldots\ldots \quad (10)$$

If R_P and L_P are the resistance and effective inductance of coil P, we may write, in the usual notation,

$$P = R_P + jwL_P,$$

and similarly

$$R = R_R + jwL_R.$$

The arms Q and S will usually consist of coils of resistance $R_Q R_S$ and inductances $L_Q L_S$, shunted by capacities $C_Q C_S$, in order to obtain the balance. Thus, to a first approximation

$$Q = R_Q + jw\,(L_Q - C_Q R_Q^2),$$
$$S = R_S + jw\,(L_S - C_S R_S^2).$$

We have also

$$p_Q = r_Q + jwl_Q,$$
$$p_S = r_S + jwl_S,$$

and since m_P and m_R represent practically pure mutual inductances,

$$m_P = jwM_P,$$
$$m_R = jwM_R.$$

Q′ and S′ in equation (9) are evidently given by

$$Q' = R_Q + r_Q + jw\,(L_Q + l_Q + M_P - C_Q R_Q^2),$$
$$S' = R_S + r_S + jw\,(L_S + l_S + M_R - C_S R_S^2).$$

Now, PRQ′S′ represent impedances of very small phase angle. Thus, to a sufficient approximation for all practical purposes we may write

$$\theta_P = \tan\theta_P = L_P w/R_P = \tau_P w,$$

where τ_P is the time constant of P, and so on. Thus equation (10) may be written

$$L_P/R_P - L_R/R_R = (L_Q + l_Q + M_P)/(R_Q + r_Q) - (L_S + l_S + M_R)/(R_S + r_S) - C_Q R_Q^2/(R_Q + r_Q) + C_S R_S^2/(R_S + r_S),$$

or, since the leads resistances r_Q and r_S are negligible compared with R_Q and R_S,

$$L_P/R_P - L_R/R_R = L_Q/R_Q - L_S/R_S + C_S R_S - C_Q R_Q + \{(l_Q + M_P)/R_Q - (l_S + M_R)/R_S\}. \quad (11)$$

This is the equation to be used for determining L_P/R_P. The quantities L_R/R_R, L_Q/R_Q, L_S/R_S, $C_S R_S$, $C_Q R_Q$, are usually known, while the term in brackets may be regarded as a correction term depending on the self and mutual inductances of the leads.

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This correction term is usually very small, and may be neglected in practice, except for work of the highest precision. In such cases it may be evaluated as follows. Suppose that the conditions of simultaneous balance have been obtained as previously described. Let C_S be the reading of the condenser shunting the arm S. Now transfer the current supply leads from the points (1) (2) to the points (3) (4) (Fig. 3), and, the link d being removed, again balance by varying the arm S. Let C_S' be the new reading of the S condenser. The value of R_S will only need altering by an amount which is negligible from the point of view of phase angle. The original condition of balance is given by (8). If the mesh currents i_3, i_4 become i_5, i_6 in the final balance, we have, as the equations for balance,

$$(i_5 + i_6)(P + p_Q + p_\alpha + 2m_P + \alpha) + i_6 Q = 0,$$
$$(i_5 + i_6)(R + p_S + p_\beta + 2m_R + \beta) + i_6 S' = 0,$$

giving

$$(P + \alpha + p_Q + p_\alpha + 2m_P)/(R + \beta + p_S + p_\beta + 2m_R) = Q/S', \quad . \; . \; . \quad (12)$$

as against the original condition,

$$(P + \alpha + p_\alpha + m_P)/(R + \beta + p_\beta + m_R) = (Q + p_Q + m_P)/(S + p_S + m_R). \; . \quad (8)$$

Now, each of these equations (8) and (12) involves a phase-angle equation of the same form as (10). We may write down by inspection the difference of these two phase-angle or time-constant equations, remembering that the lead resistances are negligible compared with the coil resistances, as

$$(l_Q + M_P)/(R_P + R_\alpha) - (l_S + M_R)/(R_R + R_\beta)$$
$$= -(l_Q + M_P)/R_Q + (l_S + M_R)/R_S + (C_S' - C_S) R_S,$$

or,

$$\frac{(l_Q + M_P)}{R_Q}\left(1 + \frac{R_Q}{R_P + R_\alpha}\right) - \frac{(l_S + M_R)}{R_S}\left(1 + \frac{R_S}{R_R + R_\beta}\right) = (C_S' - C_S) R_S.$$

Now to a first approximation, the resistances obey the usual relation for the Kelvin double bridge, viz.,

$$R_P/R_R = R_Q/R_S = R_\alpha/R_\beta.$$

Hence we have

$$\{(l_Q + M_P)/R_Q - (l_S + M_R)/R_S\}\{1 + R_Q/(R_P + R_\alpha)\} = (C_S' - C_S) R_S.$$

Thus, the correction term required is given by

$$\{(l_Q + M_P)/R_Q - (l_S + M_R)/R_S\} = (R_P + R_\alpha) \times (C_S' - C_S) R_S/(R_P + R_Q + R_\alpha). \quad (13)$$

In practice R_P is usually small compared with R_Q and R_α, which are equal. We then have the very simple relation

$$\text{Correction} = (C_S' - C_S) R_S/2. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

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Practical Details and Procedure.

It is evident from equations (11) and (13) that we have a means of finding the difference of time constant of two four-terminal shunts P and R, in terms of the time constants of the coils Q and S, and the values of their resistances and the capacities shunting them, when the above conditions are fulfilled. For Q and S we can use ordinary non-inductive ratio arms, of the order of 100 ohms and 1,000 ohms. Their time constants are likely to be of the order 10^{-8}, and the difference between them can be measured, if necessary, with an error of about 10^{-9}. Thus, since a standard shunt is not likely to have a time constant much less than 10^{-7}, the values of the time constants of Q and S are not usually of great importance.

The actual arrangement used is shown in Fig. 4. The two resistances P and R are connected in series to the current supply by means of their "current terminals." The go and return connecting leads are kept as close together as possible, so as to avoid any

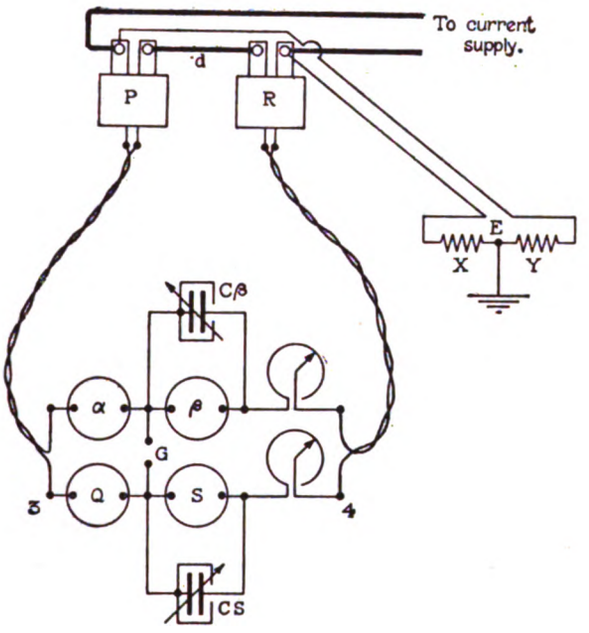

Fig. 4.

strong magnetic field. The ratio coils Q, S, α, β, are shielded coils of the type used in ordinary alternating current bridge work. Generally a ratio of 100 ohms : 1,000 ohms was used. These coils were placed some distance (say 1 metre) away from the resistances P and R, so that the possibility of any undesirable stray field effects from the main circuit is further reduced. The potential terminals of P and R are connected to the ratio arms by means of twinned flexible leads as shown. In some cases these leads were 2 or 3 metres long, and it is on this account that the correction for their self and mutual inductances was evaluated. Since it is necessary to vary the resistances of the

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ratio arms, small slide wire resistances were included in β and S or α and Q. In most of the experiments made, current of telephonic frequency was used and a telephone receiver was used as the detector G. This is convenient, since it allows the necessary sensitivity for time constant to be obtained with a smaller current than is required at lower frequencies. The bridge is, however, independent of frequency to a first approximation, and measurements were also made at a frequency of 50 cycles, in which case a vibration galvanometer was used as detector. The phase-angle adjustments were made by means of two variable air condensers, C_S and C_β, in Fig. 4. These had maximum values of about 0.002 μF. In some cases their ranges had to be extended by the use of fixed mica condensers in parallel with them. In order to locate definitely the potentials of the various coils with respect to earth, the outer current terminals of P and R were shunted by two resistances X and Y in series, their common point E being earth-connected. The values of X and Y were chosen to satisfy approximately the relations $X/Y = P/R = Q/S = \alpha/\beta$. Thus, when the bridge is balanced, the common points of PR, $\alpha\beta$, QS will be approximately at the same potential as E. The detector G will, therefore, be at earth potential, and no undesirable earth capacity effects are likely to be introduced. The shields of the condensers C_β and C_S were connected to the galvanometer side of β and S respectively. The shields of the coils $\alpha\beta$QS were earth-connected. Under these conditions, the balance point of the bridge is very stable, and the bridge very easy to work. Reversing the connexions to the current supply and the detector causes no change in the balance point.

The procedure was as follows: The bridge was connected up as in Fig. 4, the link *d* being made as low in resistance as possible. Balance was obtained by varying the resistance and capacity of the S arm. The link *d* was then opened, and the bridge again balanced by varying the resistance and capacity of the β arm. The link *d* was then again joined up and balance once more obtained by varying the S arm. Usually the adjustment required at this stage was only very slight, and the bridge was then found to be balanced when the link was either closed or open. Current was now applied to the bridge at the points (3) and (4) of Fig. 4, the link *d* being open, and balance again obtained by varying the S arm. The amount of adjustment required for this was only very small. As explained above, it represents the effect of the self and mutual inductance of the leads running up to the potential terminals of P and R. If C_S is the reading of the S condenser for the first simultaneous balance, and C_S' is the final reading of this condenser, we have from equations (11) and (13)

$$L_P/R_P - L_R/R_R = L_Q/R_Q - L_S/R_S + C_SR_S - C_QR_Q + (R_P + R_a) \times (C_S' - C_S)\, R_S/(R_P + R_Q + R_a). \quad . \quad . \quad . \quad (15)$$

In general, it is not necessary to have condensers across both the Q and S arms, and the term C_QR_Q is thus usually zero. The quantity $(L_Q/R_Q - L_S/R_S)$ may be regarded as a small correction δ_{QS}, which is known, or which can be measured by any of the usual

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methods. Further, R_P is usually negligible compared with R_Q and R_a, which are equal. The working equation, therefore, becomes

$$L_P/R_P - L_R/R_R = C_S R_S + \delta_{QS} + \tfrac{1}{2}(C_S' - C_S)R_S,$$
$$= \tfrac{1}{2}(C_S + C_S') + \delta_{QS}.$$

The points (3) (4) (Fig. 4) should, of course, be brought as near together as possible, so that current can be brought to them through bifilar twisted leads, which will induce no stray E.M.F.'s into the coils $\alpha\beta QS$. It is important to make the link d as low in resistance as possible, for the smaller this resistance compared with P and R, the less is the effect on the balance point of the values of α and β, and, therefore, the easier it is to obtain the condition of simultaneous balance.

The Primary Standard.

It will be observed that the bridge method can only give the difference of the time constants of two shunts. Thus, it is necessary to have at least one shunt of known time constant before values can be obtained for others. The ultimate standard used in these experiments was a 1-ohm four-terminal resistance, the effective inductance of which was calculable from its dimensions. This was designed by Mr. S. BUTTERWORTH, who has kindly supplied the following particulars. The resistance consists of a rectangular loop of Eureka wire C_1STUVC_2 (Fig. 5), C_1 and C_2 being the current terminals. Two brass contact pieces X and Y make contact with this loop at the potential points S and V. From the contact pieces X and Y potential leads proceed parallel to the sides of the main resistance loop, to the potential terminals P_1 and P_2. The various wires are located by passing through holes in the ebonite spacing pieces shown in Fig. 5. The current and potential terminals are as far apart as possible, so that the pairs of leads connecting them to the measuring circuit will not induce E.M.F.'s into one another. In order to calculate the effective inductance, we have to evaluate

(*a*) The self-inductance of the rectangular loop STUV, between the potential points S and V.

(*b*) The mutual inductance between the potential leads XW, YZ and the sides of the loop STUV.

(*c*) The small correction terms due to the extensions C_1S, C_2V, WP_1, and ZP_2 of the leads.

If the dimensions are as shown in Fig. 5 then, neglecting the effect of the leads (*c*), the inductance is given by the formula

$$L = 4B \log_e \{Dd/R(D + d)\},$$

R being the radius of the wire. The actual values were D = 9.68 mms.; d = 4.64 mms.; B = 10.0 cms.; R = 0.154 mm., giving L = 130 cms. = 0.130 μH. With

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the dimensions of the leads shown in Fig. 5 the correction due to these was found by calculation to be — 0.006 μH. Thus the final value for the inductance is L = 0.124 μH.

The time constants of 1-ohm resistance standards were determined by direct comparison with this calculated standard, using the method described above, with equal ratio arms, either 100 : 100 ohms, or 1,000 : 1,000 ohms. Standards of 0.1 ohm resistance were compared with a 1-ohm resistance using a ratio of 100 : 1,000 ohms; 0.01 ohm

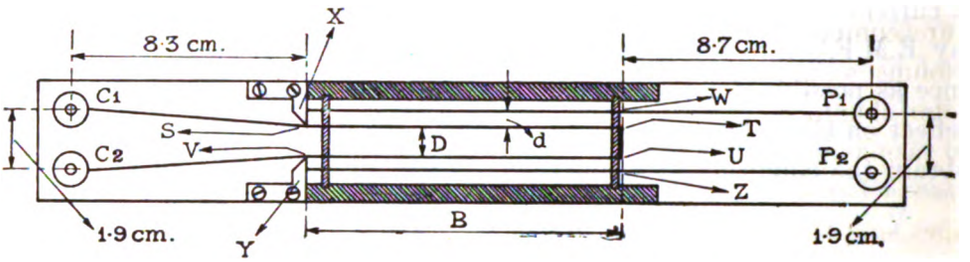

Fig. 5.

standards were compared with a 0.1 ohm standard using the same ratio, and so on down to 0.0005 ohm. Ratios other than 10 : 1 are sometimes used, but in general this is the most convenient. The accuracy in time constant obtainable on the shunts of very low resistance is of course somewhat less than that on shunts of 1 ohm and 0.1 ohm, since the errors of the various intermediate steps will be cumulative. The time constants of these low resistance shunts are, however, always higher than those of greater resistance, and it is not possible to use them with the same degree of precision, so that this point is not of great practical importance.

When no standard of calculable inductance is available, it is possible to proceed as follows. Take an ordinary 10-ohm resistance coil, and use it as a four-terminal resistance, simply by using its terminals as both current and potential terminals. Its inductance as a four-terminal resistance may be taken as practically the same as its ordinary value as a two-terminal coil, which is readily measurable. There will, of course, be small errors due to the mutual inductance between the coil and the potential leads used, but if each pair is twisted together, and they are disposed so as to make this mutual inductance small, the effect on the time constant L/R will be negligible, owing to the comparatively large value of R. A 1-ohm shunt may thus be evaluated in terms of a 10-ohm coil used in this way, and results so obtained were found to agree with those obtained by comparison with the calculated standard with an error of less than 1×10^{-8} in time constant.

Typical Results.

The time constants and effective inductances of a number of four-terminal resistances of various types have been determined by this method. The resistances varied from 1.0 ohm and 0.1 ohm for coils such as are used in alternating current bridge and potentiometer work, to 0.0005 ohm and 0.001 ohm, for shunts capable of carrying

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1,000 or 2,000 amperes, such as are used for power measurements and for testing current transformers. The results are given here, since they may be regarded as being fairly typical, and as indicating therefore the probable order of error involved in using apparatus of the particular type in question.

The measurements were nearly all made with current of frequency 800 cycles per second. The use of high frequency increases the value of the quadrature component of voltage relatively to the in-phase component, and thus makes the measurement of this quadrature component in a sense easier. In making measurements with shunts of 1 and 0.1 ohms, a current of 0.25 ampere gave sufficient sensitivity. For the 0.001 and 0.0005 ohm shunts, a current of 5 amperes was required. The detector was an ordinary 120-ohm telephone receiver of the Bell type.

The heavy-current resistance standards are used almost exclusively at power frequencies, and thus the value of the inductance at 50 cycles per second is the value actually required. It was found, however, that for all the resistances tested, the inductance is sensibly independent of the frequency up to about 1,000 cycles per second. It is an advantage of the bridge that the balance point is practically independent of the frequency, and thus to find out whether the inductance varies appreciably with the frequency, it is only necessary to raise the frequency from, say, 500 cycles per second to 800 cycles per second, and to note the change of balance point. In general the resistances are found to change slightly, but not so the inductances.

The possibility of the effective inductance changing with varying current strengths must also be considered. Any change in the inductance as the current strength is increased can only be due to one of two things, viz. :—

(*a*) A change in the distribution of the lines of flow of the current ;

(*b*) A change in the geometrical configuration of the shunt. Such changes will not occur in a well-designed shunt. Changes in the distribution of the lines of current flow will, of course, give rise also to changes of resistance, and these are easily detected. Bending and warping of the resistance alloy does occur in some cases, due to the heating effect of the current, but such shunts cannot be regarded as standards, from the point of view of phase angle, since the value of the phase angle will always be subject to considerable uncertainty. The inductance of a well-designed shunt must therefore be independent of the current strength. The method does, of course, present the advantage that it may be used with almost any desired value of current. It is merely necessary to obtain a shunt capable of use as a comparison standard, which will carry the maximum current required without distortion.

As an illustration of the order of the quantities involved in these measurements, the following observations, made on a bridge set up for the comparison of two shunts of resistances 0.1 and 1 ohm respectively, are given :

$R_P = 0.1$ ohm.	$R_Q = 100$ ohms.	$R_\alpha = 50$ ohms.
$R_R = 1.0$,,	$R_S = 1{,}000$,,	$R_\beta = 500$,,

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The bridge arms S and β were provided with small slide-wire adjustments of resistance, and were shunted with variable air condensers. When the first simultaneous balance was obtained, the reading of the condenser C_S was 1,330 $\mu\mu$F $= 1{,}330 \times 10^{-12}$ farad. The current supply leads were now transferred to the points (3) (4), Fig. 3, the link *d* removed, and balance again obtained by adjusting the arm S. The new reading of the condenser was $C_S' = 1{,}342$ $\mu\mu$F. The readings of the various resistances, R_Q, R_S, etc., were not noted in detail, since they were always within about 0.5 per cent. of their nominal values, and thus their actual values are of no consequence from the point of view of inductance measurement. Similarly, the reading of the condenser across the arm β is not required. The time constants of the ratio arms Q and S were known from two-terminal measurements to be 0.6×10^{-8} and 1.3×10^{-8} respectively, thus giving rise to the correction term $\delta_{QS} = L_Q/R_Q - L_S/R_S = -0.7 \times 10^{-8}$. The time constant of the 1-ohm standard R had been found by comparison with the calculated standard to be $L_R/R_R = 1.6 \times 10^{-8}$. Thus, from equation (14) we find

$$L_P/R_P - 1.6 \times 10^{-8} = -0.7 \times 10^{-8} + 133._0 \times 10^{-8} + \tfrac{1}{3} \times 12 \times 10^{-9},$$

giving $L_P/R_P = 134._3 \times 10^{-8}$, and therefore $L_P = 0.134$ μH. The various correction terms are seen to be almost negligible. The values obtained for a series of typical shunts are given in Table I.

TABLE I.—Effective Inductances of Typical Shunts.

Resistance (ohm).	Type.	Maximum current (amps.).	Time constant (secs.).	Effective inductance (μH).
1·0	N.P.L.	0·5	$1{\cdot}6 \times 10^{-8}$	$0{\cdot}016_0$
1·0	Drysdale-Tinsley	3	$7{\cdot}5 \times 10^{-7}$	$0{\cdot}75_0$
0·1	,,	22	$13{\cdot}4 \times 10^{-7}$	$0{\cdot}134_0$
1·0	N.P.L. box	2	$2{\cdot}3_1 \times 10^{-7}$	$0{\cdot}231_0$
0·1	,,	20	$2{\cdot}3 \times 10^{-7}$	$0{\cdot}023_0$
0·04	,,	50	$2{\cdot}0 \times 10^{-7}$	$0{\cdot}008_0$
0·02	N.P.L. tube	100	$2{\cdot}9 \times 10^{-7}$	$0{\cdot}005_8$
0·01	,,	200	$4{\cdot}1 \times 10^{-7}$	$0{\cdot}004_1$
0·004	,,	500	$15{\cdot}5 \times 10^{-7}$	$0{\cdot}006_2$
0·002	,,	1,000	11×10^{-7}	$0{\cdot}002_2$
0·001	,,	2,000	32×10^{-7}	$0{\cdot}003_2$
0·005	Drysdale-Tinsley	200	$2{\cdot}2 \times 10^{-7}$	$0{\cdot}001_1$
0·001	,,	700	14×10^{-7}	$0{\cdot}001_4$
0·0005	,,	1,000	25×10^{-7}	$0{\cdot}001_3$

The shunts described in Table I as being of " N.P.L. box " type are air-cooled shunts consisting of a number of bifilar loops in parallel. As is to be expected, the time constant is practically independent of the actual resistance, since all the loops are exactly similar, and the required resistance is obtained simply by putting an appropriate number in

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parallel. The " N.P.L. tube " type of shunt is the Paterson-Rayner* water-cooled tubular shunt.

The inductances of some of these shunts can be calculated approximately from their dimensions, *e.g.*, the Drysdale-Tinsley 0.001 ohm shunt consisted of two concentric cylinders, joined together at one end. The current is led in and out at the other end by " current leads " consisting of a pair of copper plates, parallel and as near together as possible so that they will produce practically no stray magnetic field, and the potential leads are very short, so that the mutual inductive effect associated with them is practically negligible. The mean diameter of the cylinder was 16 cms. = 2 R, say. Their length (l) was 15 cms. and the radial distance between them (t) was 0.35 cm. If a current I flows down the inner tube and up the outer one, then evidently the field is zero everywhere except in the annular space between the tubes. The mean field intensity in this region is $2I/R$, and thus the total flux is $2Ilt/R$. The inductance, or flux per unit current, is thus approximately given by $L = 2lt/R = 1.3$ cm. $= 0.0013$ μH. The measured value was 0.0014 μH. A few calculated values given by PATERSON and RAYNER for their water-cooled shunts are given in Table II, together with the corresponding observed values.

TABLE II.—Observed and Calculated Values of Inductance for Paterson-Rayner Shunts.

Shunt.	Inductance.	
	Calculated (cms.).	Observed (cms.).
0·02 ohm	5·4	5·8
0·01 ,,	3·4	4·1
0·002 ,,	3·7	2·2
0·001 ,,	3·0	3·2

In these cases the agreement between the two values is not perfect, although it would be near enough for many practical purposes. The differences are probably due partly to the fact that the conditions required by the formula used for the calculations are only imperfectly realised in the shunts (there is in practice a gap in the middle of the potential sheath), and partly to the fact that the insulation between the tube and sheath has been renewed since the calculations were made, and that its thickness is now almost certainly somewhat different from its original value. There is little doubt that the inductances of such shunts can now be measured with an accuracy greater than that obtainable by approximate calculations.

* C. C. PATERSON and E. H. RAYNER, 'N.P.L. Collected Researches,' Vol. 6, Paper 9, p. 99; or 'J. Inst. Elec. Eng.,' Vol. 42, p. 455, 1909.

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XII.

A CONTACT THEORY OF DIELECTRIC ABSORPTION AND POWER LOSSES

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ABSTRACT.

A simple explanation is proposed of absorption, residual charge, and allied phenomena, in solid dielectrics. In accordance with classical theory, the dielectric is assumed to possess a certain true capacity, defined by its dielectric constant, and a certain conductance, here considered as probably electrolytic in type. The so-called anomalous properties are considered to be due to the properties of the contact surfaces of the dielectric and its metal electrodes. These contact surfaces appear to offer great resistance to the passage of ions or electrons across them, so that when an E.M.F. is applied to a condenser there is an accumulation of charges at the surfaces. These charges constitute the absorbed and residual charges. The experimental evidence in favour of this hypothesis is reviewed.

The behaviour of each contact surface is such that it may be represented by a large capacity in parallel with a high resistance, and thus a capacity-resistance combination is suggested, which is equivalent to an actual condenser. The properties of this combination are considered mathematically with reference to the experimental laws of absorption and power losses in alternating fields, and the extent to which the theory is capable of explaining the experimental results is pointed out. The bearing of the theory on dielectric measurements under various conditions is also discussed.

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A CONTACT THEORY OF DIELECTRIC ABSORPTION AND POWER LOSSES.

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I.—*Introductory.*

Some time ago Dr. D. W. Dye, F.R.S., and the author made a series of measurements on the dielectric properties of insulating materials in thin sheet form, using two different methods: (1) The change in capacity produced on inserting the material in the air gap between two parallel metal plates was measured; (2) Mercury electrodes were applied to the two sides of the sheet under test, and the capacity of the condenser so formed was measured. Values of the dielectric constants of the materials calculated from the results of these observations by the usual formulæ showed systematic discrepancies, the mercury electrode method always giving the higher result. In order to explain these discrepancies, the writer put forward the suggestion that there was a contact effect at the mercury dielectric surface, *e.g.*, a "polarisation capacity" effect such as occurs in electrolytes. We have considerable reason to believe that the discrepancies quoted in our paper on mica* are largely due to slight irregularities in the construction of the air gap; but experiments made on varnished cloth did not seem to be sufficiently explained by these irregularities, and led to the examination of the possibilities of the contact effect suggested above. It now appears that such an effect is capable of explaining many of the anomalous properties shown by dielectrics, and it is proposed to show in this paper how the phenomena of the polarisation of dielectrics under a constant electrostatic field (*i.e.*, the so-called absorption phenomena), the residual charge, and the power loss in alternating fields are all capable of explanation in terms of conduction, electrolytic in type, in the body of the insulating material, associated with what may be described as a contact capacity effect at the surface.

II.—*Polarisation in Constant Fields.*

According to Maxwell's theory, the electric actions occurring in any material are characterised by two constants, the specific conductivity of the material and its specific inductive capacity or dielectric constant. If an electrostatic field of strength F is applied to a substance, electric displacement of two kinds is produced:—

(*a*) There is a continuous movement of electricity in the direction of the applied field. If σ is the specific conductivity of the material, then the quantity of electricity

* Dye and Hartshorn, 'Proc. Phys. Soc. Lond.,' Vol. 37, p. 42, 1924.

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crossing each unit of area of the substance (the area being measured at right angles to the flow) in each unit of time is given by $F\sigma$. This displacement of electricity continues at this constant rate as long as the field F is maintained.

(*b*) Another kind of electric displacement is also produced. This does not increase indefinitely with time, but almost instantly reaches its final value of $KF/4\pi$, where K is the dielectric constant of the material.

At the present time we associate the electric displacement of the first kind with the free electrons in the material, and that of the second type with the electrons which are bound by " elastic " forces to the fixed nuclei of the atoms or molecules.

It is a matter of common knowledge, however, that when a condenser is made by applying metal electrodes to any dielectric which is partly or wholly solid, and an E.M.F. is applied to these electrodes so as to maintain a constant potential difference between them, the actual electric displacement occurring is not simply that indicated by this simple theory. What is actually observed is—

(1) A comparatively large instantaneous* electric displacement. This is considered to be the Maxwellian displacement $KF/4\pi$.

(2) A further electric displacement, not at a uniform rate such as would identify it as the displacement due to conductance $\int F\sigma dt$, but at a rate which decreases with time, rapidly at first but afterwards more slowly, until finally (after many hours or even months) the rate of displacement becomes very small and constant, so that it is then assumed to be the conductance displacement, and is used to determine the constant σ.

In other words, what is actually observed is—

(1) The true capacity charge, regarded as instantaneous.

(2) The " absorption " current, which gradually diminishes to zero, leaving only

(3) The leakage current.

The simple classical theory further says that when the field F is removed the displacement $KF/4\pi$ vanishes, and the conductance displacement ceases, *i.e.*, there is an instantaneous displacement $-KF/4\pi$, which is the collapsing of the capacity displacement caused by the initial application of the field, and is thus in the opposite direction to the original displacement, and no further conductance displacement takes place. Thus the displacement due to capacity is regarded as reversible; that due to conductance is not. In practice, if the condenser which we have imagined to be charged is

* Following the usual custom in discussing dielectric anomalies, the actual electric displacement is regarded as consisting of two portions, one being regarded as instantaneous,and the other being a function of time. This division is, of course, quite arbitrary, and is simply adopted as a matter of convenience. An exact specification of the " instantaneous displacement " is probably impossible. A masterly discussion of the matter has been given by HOPKINSON, ' Phil. Trans. Roy. Soc.,' A, Vol. 189, p. 109, 1897.

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short-circuited by connecting its plates together by means of a wire, what is observed is—

(1) An instantaneous displacement equal in magnitude to the original instantaneous displacement, but of opposite sign.

(2) A further displacement also in the opposite direction to the original displacement. This occurs at a rate which is rapid at first, but which gradually decreases down to zero.

In other words, we observe the true capacity discharge, regarded as instantaneous, followed by the residual discharge current, which gradually decreases down to zero, leaving the condenser entirely uncharged. If, however, the condenser is open-circuited (*i.e.*, the wire connecting its plates together is removed) before the discharge is complete (and this may take hours or days), a charge gradually builds up on the electrodes or plates of the condenser. This is known as the residual charge, and is of the opposite sign to the charge taken by the condenser during the time the voltage is applied.

Several theories have been put forward in order to explain these anomalous charging and discharging currents in a dielectric, *i.e.*, currents or electric displacements which are apparently independent of the conductance σ and the dielectric constant K. MAXWELL showed that they can be explained without any extension of the simple theory if the dielectric be assumed to consist of layers of materials of different properties, so that σ/K is different for different layers. This explains the existence of the phenomena in the case of a cable with a laminated dielectric; but H. A. WILSON,* THORNTON† and S. W. RICHARDSON‡ have found the same effects in crystalline quartz and iceland spar, and it is difficult to believe that such substances are not of homogeneous structure in any particular direction. VON SCHWEIDLER§ has advanced the theory that a dielectric possesses, in addition to the displacement electrons, which are capable of vibratory motion about their equilibrium position with a definite natural period and definite damping, other ions in which the damping is so great that their motion is aperiodic. These ions move very slowly under the influence of an applied field, and thus give rise to the anomalous charging current. The anomalous discharge and the residual charge are of course considered to be due to their slow return to their original equilibrium positions. Unlike MAXWELL'S theory, this assumes that the dielectric actually possesses other properties than those characterised by σ and K, a complication which one would prefer to avoid if possible. THORNTON† has supposed that the anomalous displacement is due to a further motion of the displacement electrons, due to inter-attractive forces consequent upon their initial displacement by the applied field. It is, however, difficult

* H. A. WILSON, 'Proc. Roy. Soc.,' A, Vol. 82, p. 409, 1909.

† W. H. THORNTON, 'Proc. Phys. Soc. Lond.,' Vol. 22, p. 186, 1910.

‡ S. W. RICHARDSON, 'Proc. Roy. Soc.,' A, Vol. 92, pp. 41, 101, 1916.

§ VON SCHWEIDLER, 'Ann. der Phys.,' Vol. 24, p. 711, 1907.

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to see why this further displacement should take place so slowly, and why it should exceed in amount the displacement predicted by the Lorentz theory of dielectrics, which it must do if it is to explain the observed facts.

The theory now proposed is as follows: The dielectric is assumed to possess the two properties defined by K and σ, only. Thus it possesses a definite capacity and a definite conductance. The conductance is considered to be electrolytic in type. In Fig. 1 let AB represent two metal electrodes applied to a slab of dielectric. In practice the metal will probably never actually make true contact with the dielectric, such contact as one gets in a soldered joint, for example. In the diagram, therefore, a small gap is shown between the dielectric faces CD and the metal faces AB. Now suppose A and B are connected to a battery so as to maintain a constant potential difference between them. The first consequence is that a potential gradient is set up in the dielectric and thus a definite displacement proportional to the dielectric constant of the material K is produced. This is the normal charge. Now, as a consequence of the potential gradient in the dielectric, the ions begin to move so as to produce a current proportional to its conductance. The positive ions move towards C, say, and the negative ones towards D. Corresponding to this current in the dielectric, there must of course be a movement of free electrons in the connecting leads to A and B. Thus we have a further accumulation of negative electrons at A, whilst positive charges preponderate still further at B, as shown. The theory now supposes that these ions at the contact faces find great difficulty in crossing over from dielectric to metal, and opposite charges continue to accumulate. It is obvious that these charges at C and D cause a back E.M.F. in the dielectric, which tends to stop this current. Thus this so-called anomalous current, or absorption current, gradually gets smaller and smaller, until finally the number of ions moving up to C and D is only equal to the numbers which can cross the contact face. This is the current which is usually regarded as the true leakage current. According to this theory, when the condenser has reached this steady state (and thus from the usual point of view is fully charged) the actual potential gradient in the dielectric is comparatively small, and therefore the true dielectric displacement is small. The charge on the contact faces may, however, be very great, and this is considered to be the "absorbed charge." The voltage distribution in this case is shown in Fig. 1 (*b*).

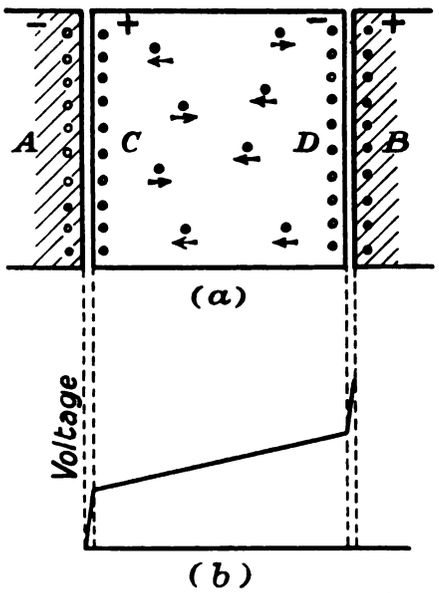

FIG. 1. Diagrammatic Representation of a Dielectric with Metal Electrodes.

Consider now the phenomena of discharge. Let the plates AB be connected by a wire so as to equalise their potentials. Let V be the voltage applied to A and B in the

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first case, and let x be the thickness of the dielectric, and $x + \delta x$ the effective distance between A and B, *i.e.*, the effective thickness of the dielectric when allowance is made for the imperfect electrode contact. When the condenser is charged, let P be the back E.M.F. due to the accumulation of ions round C and D, then the resultant voltage gradient in the dielectric is $\frac{V}{x+\delta x} - \frac{P}{x}$, and the true dielectric displacement is $\frac{K}{4\pi}\left(\frac{V}{x+\delta x} - \frac{P}{x}\right)$. On short-circuiting, V is reduced to zero, and thus the voltage gradient becomes $-\frac{P}{x}$, and the true dielectric displacement $-\frac{K}{4\pi} \cdot \frac{P}{x}$. Thus the change of displacement on short-circuiting is $-\frac{K}{4\pi}\left(\frac{V}{x+\delta x}\right)$. This is seen to be equal to the instantaneous charge on applying the voltage V, and thus agrees with experiment. The surprising conclusion arrived at, however, is that this instantaneous discharge, although equal to the instantaneous charge, is not merely the collapsing of this charge. In fact, instead of becoming discharged on short-circuiting, the dielectric becomes charged or polarised in the opposite direction by the back E.M.F. due to the ions at the contact surfaces. Another effect of this E.M.F. is that now the ions begin to move in the opposite direction, thereby causing the " anomalous " discharge current. The greater the extent of this motion the smaller becomes the back E.M.F. of the ions, and thus this current gradually decreases down to zero. If the external circuit is not kept closed, the movement of the ions away from the contact surfaces liberates corresponding charges in the electrodes, and as these charges cannot flow along the connecting wire and so neutralise each other, there is an accumulation of electricity on the condenser plates. This is the residual charge. It may be large enough to raise the condenser plates to quite high voltages.

CURIE* and S. W. RICHARDSON† have discussed their observations on dielectrics in terms of an " internal E.M.F. of polarisation." On the present theory this is simply the back E.M.F. due to the accumulation of ions at the contact faces. RICHARDSON made measurements of the E.M.F. by means of a potentiometer arrangement. He found that P, the internal E.M.F. of polarisation, increased with the time of charging, rapidly at first, but afterwards more slowly, which is, of course, exactly what is to be expected by the theory.

III.—*The Equivalent Capacity-Resistance Combination.*

It is to be noted that the metal-dielectric contact surface behaves like a large condenser, and according to this ionic theory, these " contact condensers " must be supposed to possess very high resistance. Thus we may represent the actual condenser by the

* CURIE, ' Ann. Chim.'

† RICHARDSON, *loc. cit.*

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combination of perfect condensers and resistances shown in Fig. 2. Here C_1 represents the true capacity of the dielectric, and R_1 is its true resistance, C_A and C_B are the contact capacities, and R_A and R_B the contact resistances. In all dielectrics in which the "absorption current" greatly exceeds the leakage current we must suppose that the true resistance of the dielectric is much less than the contact resistances.

When this theory was first conceived, this assumption was so contrary to current notions that it appeared to be a very serious objection. It means, for example, that if two samples of the same dielectric, but of different thicknesses, are placed between metallic electrodes (tin-foil or mercury, etc.), and their resistances are measured in the usual way, then, in spite of the difference in thickness, the resistances will be very nearly the same, since nearly all the resistance lies in the contact surfaces. There seemed to be no experimental evidence for this, and all determinations of the specific resistance of dielectrics tacitly assume that such is not the case. Quite recently, however, H. M. Barlow,* following a totally different line of investigation, has arrived at the same conclusion—at least for certain insulating materials used in practice. Barlow was investigating the frictional forces between sliding surfaces across which a direct current was flowing. He found that when one of the substances in contact was a metal, and the other a comparatively poor insulator, such as slate, celluloid, red fibre, or lithographic stone, the passage of a current across the contact caused a considerable increase in the frictional force between the surfaces. He traced this to an increase in the normal pressure between the surfaces due to the fact that nearly the whole of the potential drop between the two substances was located at the contact surface, *i.e.*, the contact resistance was enormously greater than the body resistance. In other words, the increase of normal pressure is due to the attractive force between opposite charges which accumulate on either side of the contact surface, exactly as the present theory indicates. These poor insulators naturally show the effect to a much greater extent than the best dielectrics. Barlow did not detect it in the case of glass, ebonite, or micanite; but in such cases the surface charges would take hours to reach anything approaching their maximum values, and the effect might very easily escape detection on this account. Barlow also quotes some results of Addenbrooke† for the direct-current resistivity ρ of celluloid obtained by measurements on sheets of different thickness, using mercury electrodes. Different values are obtained for different thicknesses, in fact ρ is very nearly inversely proportional to the thickness, which again points to the fact that the

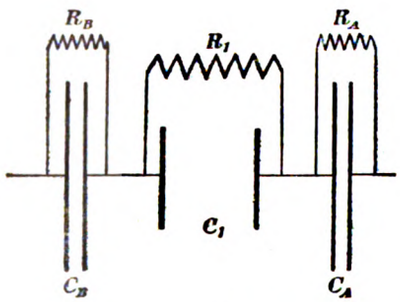

Fig. 2. Network Equivalent to an Imperfect Condenser.

* H. M. Barlow, 'J. Inst. Elec. Eng.,' Vol. 62, p. 133, 1924.

† Addenbrooke, 'J. Inst. Elec. Eng.,' Vol. 57, p. 13, 1919; 'Proc. Phys. Soc. Lond.,' Vol. 27, p. 291, 1915.

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bulk of the resistance is in the contact face. BARLOW himself made measurements of a similar nature on slate and lithographic stone, with the same result. Further, the work of RICHARDS* shows that charges of opposite sign can exist on either side of a contact surface of metal and dielectric, even when these surfaces make wringing contact, each surface being flat to half a wave-length of sodium light. RICHARDS measured these charges in the case of contacts of glass, quartz, fluorite and ebonite with steel. (In this case the charges were produced merely by wringing the surfaces together, and thus they may be of a different nature from those produced as a result of the application of an external E.M.F.)

In view of all this, we see that the assumptions required by the present theory are not without some experimental foundation.

IV.—*Some Practical Considerations.*

It has been remarked that the contact surface of metal and dielectric may be regarded as acting like a large condenser of high resistance, and that in practice a condenser may be represented by the combination of ideal capacities and resistances shown in Fig. 2. Consider now some of the simplest properties of the system, and it will be found that many obscure phenomena met with in practice are easily explained.

(*a*) *A.C. and D.C. Resistance.*—It is a well-known fact that the alternating-current resistance of a dielectric is enormously less than the direct-current value. By reference to Fig. 2 it is at once seen that the D.C. resistance is $R_1 + R_A + R_B$, since the final constant leakage current must traverse these three resistances. If now alternating voltage is applied, the current can cross the surfaces as a capacity current through the condensers C_A and C_B, and since their impedances at the frequencies met with in practice are much less than those of R_A and R_B, the current (r.m.s.) passing in this case for a given applied voltage (r.m.s.) is much greater than the current flowing when unidirectional voltage of the same magnitude is applied. This current now divides between R_1 and C_1, and since R_1 is much less than R_A, the portion passing through R_1 is quite considerable, the actual amount depending on the frequency. Thus the power dissipated in the resistance R_1, with a given applied voltage, is far greater for A.C. than for D.C. Thus may arise the so-called dielectric hysteresis loss, as a consequence of which the effective A.C. resistance is much less than the effective D.C. resistance, though, according to the present theory, if the true resistance R_1 could be measured it would be the same in both cases.

(*b*) *The Absorbed Charge.*—When measurements of the so-called absorbed charge are made, the most astonishing point is its enormously high value, *e.g.*, in an experiment made by the writer, a sample of varnished cloth with mercury electrodes was charged for about 48 hours with a constant voltage. The electrodes were then short-circuited,

* H. F. RICHARDS, 'Phys. Rev.,' Vol. 22, p. 122, 1923.

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and the residual discharge current measured during the next 48 hours. The total charge coming out in this time (obtained from the time-integral of the current) was such as would correspond to a dielectric constant of 700 for the sample. The measured dielectric constant at 800 cycles per sec. was 4·7. Thus the absorbed charge must be hundreds of times greater than the true capacity charge. This is explained by the enormous values of the contact capacities C_A and C_B compared with C_1. When the final steady state is reached the potential drop across C_A and C_B is considerable, owing to the high values of R_A and R_B, and thus these condensers possess very large charges which are slowly released as the discharge proceeds.

V.—*Approximate Mathematical Theory.*

Before using the model of Fig. 2 for the purpose of calculation it is necessary to note the assumptions involved. It means that the back E.M.F. due to the accumulation of ions at the surfaces is proportional to the total accumulation of charge of either sign, *e.g.*., the back E.M.F. at surfaces $AC = M \int idt$, where $M = 1/C_A$ and $\int idt$ is the total charge conveyed to the surface C by the positive ions moving up to it (Fig. 1). If now the direction of the applied E.M.F. be reversed, then negative ions will move up to the surface C, and in this case we have the back E.M.F. $= M' \int idt$. The contact capacity is now $C_A' = 1/M'$. It is not necessarily the same as in the previous case, since it may depend on the size of the ions moving up to the surface, and the positive and negative ions may not be of the same size. Similarly, the new contact resistance R_A' may have a different value from R_A, since the positive and negative ions may not move with the same velocity under the action of a given force. The same thing may also occur at the other contact surface, so that when the condenser is carrying alternating current the values of the contact resistances and capacities may also alternate. As a first approximation we shall, however, make the assumption that the contact resistances and capacities are constant for varying applied E.M.F.'s, and, further, that they are the same at the two surfaces, *e.g.*, suppose we are dealing with a parallel plate condenser, the electrodes being of the same size and of the same metal, then by symmetry the two surfaces may be regarded as possessing identical properties.

In Fig. 2 let V_A be the potential differences across C_A, V_1 that across C_1, and V_B that across C_B. Let I be the total current in the circuit. Then

$$I = \frac{V_A}{R_A} + C_A \frac{dV_A}{dt} = \frac{V_1}{R_1} + C_1 \frac{dV_1}{dt} = \frac{V_B}{R_B} + C_B \frac{dV_B}{dt}.$$

Also, if V is the total potential difference across the condenser

$$V = V_A + V_1 + V_B.$$

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If now we assume $C_A = C_B = 2C_0$, and $R_A = R_B = \frac{1}{2}R_0$, then $V_A = V_B = \frac{1}{2}V_0$ (say), and the above equations reduce to

$$I = \frac{V_1}{R_1} + C_1 \frac{dV_1}{dt},$$

$$I = \frac{V_0}{R_0} + C_0 \frac{dV_0}{dt},$$

$$V = V_1 + V_0.$$

Here C_0 may be regarded as representing the total contact capacity effect, R_0 the total contact resistance effect, and V_0 is the total surface voltage drop. These equations are seen to be of exactly the same form as those given by MAXWELL'S theory of a stratified dielectric when only two layers are considered. The general solution has been given in a convenient form by MEYER.* It may be written

$$I = C\frac{dV}{dt} + \frac{V}{R} + \frac{kC}{T} e^{-\frac{t}{T}} \int_{-\infty}^{t} e^{\frac{t}{T}} \frac{dV}{dt}\, dt, \quad \ldots\ldots\ldots\ldots \quad (1)$$

where

$$C = \frac{C_1 C_0}{C_1 + C_0},$$

$$R = R_1 + R_0,$$

$$T = \frac{R_1 R_0 (C_1 + C_0)}{R_1 + R_0},$$

$$k = \frac{(R_1 C_1 - R_0 C_0)^2}{C_1 C_0 (R_1 + R_0)^2}.$$

Consider now the cases of most importance in practice.

(1) Let a constant voltage E be applied at the instant $t = 0$. Then the current is given by

$$I = \frac{E}{R} + E\frac{kC}{T} e^{-\frac{t}{T}}. \quad \ldots\ldots\ldots\ldots \quad (2)$$

It may be regarded as consisting of two parts, the leakage current $E/(R_1 + R_0)$, and the "after-effect" or absorption current $E\frac{kC}{T} e^{-\frac{t}{T}}$, which decays exponentially with time, approaching a zero value as the value of t becomes very great.

Suppose now that having kept the applied voltage constant for a time interval t_1,

* U. MEYER, 'Verh. Deutsch. Phys. Ges.,' Vol. 19, p. 139, 1917.

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it is reduced to zero by short-circuiting the condenser. In the interval $t = 0$ to $t = t_1$ the current is given by equation (2). For values of t greater than t_1 it is given by

$$\begin{aligned} I &= \frac{kC}{T} e^{-\frac{t}{T}} \left[E - E e^{\frac{t_1}{T}} \right] \\ &= \frac{EkC}{T} e^{-\frac{t}{T}} - \frac{EkC}{T} e^{\frac{(t-t_1)}{T}} \quad \ldots\ldots\ldots\ldots\ldots \quad (3) \end{aligned}$$

The first term in this expression is smaller than the second, thus the current is now in the opposite direction. Further, the relation between this residual discharge current and the anomalous charging current is exactly that stated in HOPKINSON'S "superposition principle,"* which was in agreement with his experimental results.

If the applied voltage E is kept constant for an indefinitely long time, then by integration we see that the total absorbed charge is EkC. If then the condenser is short-circuited, the residual current, if t is now measured from the instant of short circuit, is given by $-E\frac{kC}{T}e^{-\frac{t}{T}}$, *i.e.*, the residual discharge current is given by exactly the same expression as was the original absorption current, and if this current is allowed to flow for an indefinitely long time the total charge conveyed by it is $-EkC$, *i.e.*, the absorbed charge is reversible as regards quantity of electricity flowing (though not as regards energy). These relations are borne out by experiment.†

The term CdV/dt in the general expression for the current through the condenser (1) has not yet appeared in this discussion. It of course represents the instantaneous charge or discharge. C is the "geometric capacity" of the condenser. It differs slightly from the true capacity of the dielectric C_1, on account of the surface capacity C_0, but when C_0 is large (which must be the case if the electrodes make good contact with the dielectric), C and C_1 are practically identical.

With regard to the magnitude of the absorption current, we see that it is directly proportional to the applied voltage and also to kC/T. Substituting, we find that

$$\frac{kC}{T} = \frac{\left(1 - \frac{R_1 C_1}{R_0 C_0}\right)^2}{R_1 \left(1 + \frac{R_1}{R_0}\right)\left(1 + \frac{C_1}{C_0}\right)^2},$$

which reduces to $1/R_1$ when R_0 and C_0 are so large compared with R_1 and C_1 that the ratios R_1/R_0 and C_1/C_0 are negligible compared with unity.

Thus, in general, the smaller the value of R_1 the larger will be the absorption current. This accounts for the rapid increase in the absorption current with rise of temperature.

* J. HOPKINSON, 'Original Papers,' Vol. 2, Paper 19, p. 10. Cambridge Press (1901).

† LUBBEN, 'Arch. Elektrotech.,' Vol. 10, p. 283, 1921; VON SCHWEIDLER, *loc. cit.*

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(2) Suppose the applied voltage is alternating and of sine-wave form, so that it may be represented by

$$V = V_0 e^{j\omega t},$$

where frequency $= \omega/2\pi$.

Then the expression for the current becomes

$$I = V\left[j\omega C + \frac{1}{R} + j\frac{\omega kC}{1 + j\omega T}\right]$$
$$= V\left[j\omega\left(C + \frac{kC}{1 + \omega^2T^2}\right) + \frac{1}{R} + \frac{\omega^2 kCT}{1 + \omega^2T^2}\right].$$

Thus the effective capacity of the condenser at the frequency $\omega/2\pi$ is

$$C_e = C + \Delta C = C + \frac{kC}{1 + \omega^2T^2}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

where ΔC is the increase of capacity due to "absorption," and the effective conductance is

$$G = \frac{1}{R} + \frac{\omega^2 kCT}{1 + \omega^2T^2}. \quad \ldots\ldots\ldots\ldots \quad (5)$$

The loss angle δ is given by

$$\frac{C + \Delta C}{C}\tan\delta = \frac{\omega kT}{1 + \omega^2T^2} + \frac{1}{RC\omega}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

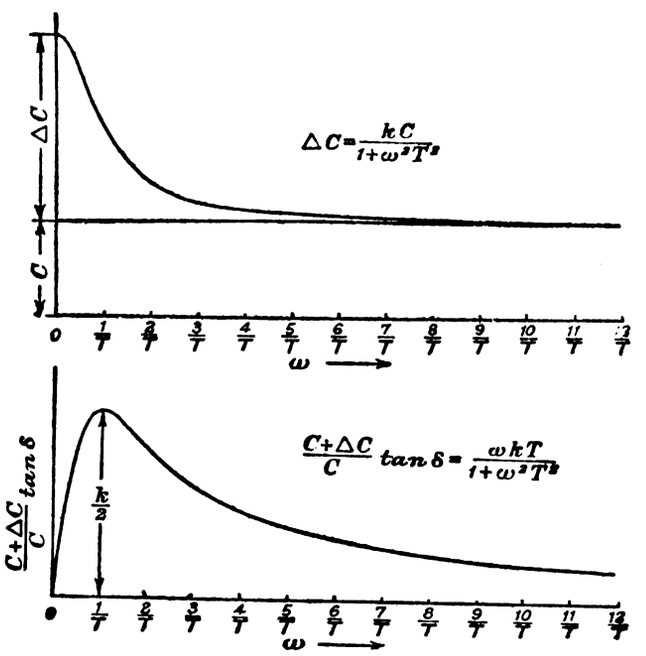

FIG. 3. Variation of Capacity and Loss Angle with Frequency.

The way in which these quantities vary with the frequency is shown in Fig. 3. The effective conductance, and therefore the power loss, is much greater for A.C. than for

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D.C., and this power loss increases rapidly as the frequency increases. The various functions are of exactly the same form as those given by MAXWELL'S theory.

VI.—*Comparison with Experimental Results.*

When the formulæ given by this approximate theory are compared with the results of experiments, it is found that, although the capacity, power loss and absorption current vary with time or frequency in a manner which may be roughly represented by the formulæ, the agreement is by no means perfect, *e.g.*, the absorption current for a constant impressed voltage decays rapidly with time, but the actual law of the decay is not a simple exponential one. Similarly, although the capacity and power factor (sin δ, which is practically the same as tan δ) in general both increase as the frequency decreases, the laws are not exactly those shown in Fig. 3. This of course may be explained by the fact that the assumptions made are known to be only approximately fulfilled. As a first approximation we have represented the contact effects by a single capacity C_0 with shunt resistance R_0. We have shown above that they could be better represented by two different capacities C_A, C_B, with corresponding resistances R_A, R_B (Fig. 2). In this case the equations are of the same form as those for MAXWELL'S theory of the stratified dielectric with three layers.* The general solution is found to be of the form

$$I = C\frac{dV}{dt} + \frac{V}{R} + \frac{k_1 C}{T_1} e^{-\frac{t}{T_1}} \int_{-\infty}^{t} e^{\frac{t}{T_1}} \frac{dV}{dt}\, dt + \frac{k_2 C}{T_2} \int_{-\infty}^{t} e^{\frac{t}{T_1}} \frac{dV}{dt}\, dt,$$

where

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_A} + \frac{1}{C_B},$$

$$R = R_1 + R_A + R_B,$$

and k_1, k_2, T_1, T_2 are constants similar to those in (1), their values depending on the values of C_1, C_A, C_B, R_1, R_A and R_B. Thus the function representing the decay of the absorption current now contains two exponential terms. The increment of capacity due to absorption is given by

$$\Delta C = \frac{k_1 C}{1 + \omega^2 T_1^2} + \frac{k_2 C}{1 + \omega^2 T_2^2},$$

whilst the expression for loss angle due to absorption is

$$\tan \delta = \frac{C}{C + \Delta C}\left\{\frac{k_1 \omega T_1}{1 + \omega^2 T_1^2} + \frac{k_2 \omega T_2}{1 + \omega^2 T_2^2}\right\}.$$

These expressions are capable of representing the experimental results with greater accuracy than the simpler expressions, though, as they are not convenient in use, they are not of much practical importance.

* U. MEYER, *loc. cit.*

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Almost all the theories which have been advanced to explain dielectric anomalies have given expressions of this form—*cf.* those of MAXWELL, PELLAT, VON SCHWEIDLER. In order to explain the observed facts, VON SCHWEIDLER* expressed the absorption current as a function of time by an infinite series of exponential terms, whilst WAGNER† showed that nearly all the observations could be explained by supposing the time constants of the exponential terms were grouped about certain most probable values, according to a distribution law based on the laws of probability. It is easy to see that the present theory would require extension on the same lines if any set of experimental results were to be explained quantitatively. The contact effects at each electrode have been represented by an ideal condenser, together with a shunt resistance. This can only be a rough approximation. The "double layer" at each electrode is not likely to be of uniform thickness, and thus the surface of the dielectric itself will not be an equi-potential surface. Surface conduction will therefore play a small part at least. Further, each element of the contact surface must be regarded as a small condenser with its appropriate resistance, the values changing from point to point over the surface. Each element may give rise to an exponential term in the absorption current-time function, and thus this function may be regarded as consisting of a series of exponential terms of varying time constant, the actual value of the time constant depending on the particular element of contact surface concerned. Further, it is quite probable that at any one electrode there would be a certain most probable value of contact capacity and resistance for any element of the contact surface. At the other electrode there might be a similar probable value. Thus the exponential terms in the expression for absorption current might be expected to be grouped about one or two probable values.

WAGNER‡ has shown that functions of this form are capable of representing observations he has made on a very wide range of insulating materials. The expressions are of course very complicated, and not at all suitable for calculation in practice. They will not, therefore, be reproduced here. Their value merely lies in the fact that they show how the very complicated results observed in practice may be explained in terms of a theory which in its physical aspect is comparatively simple.

VII.—*The Air Gap and Mercury Electrode Methods.*

Experiments on dielectrics are usually made under one of two conditions :—

(1) Metal electrodes are pressed into contact with the dielectric ; or

(2) The dielectric is placed in an air gap between two electrodes.

It may be worth while to examine the two methods in the case of most practical importance (alternating current), in the light of the present theory. In each case we have the dielectric, which we shall consider as a capacity C_1 with shunt resistance R_1,

* VON SCHWEIDLER, *loc. cit.*

† K. W. WAGNER, 'Ann. der Phys.,' Vol. 40, p. 817, 1913.

‡ K. W. WAGNER, 'Arch. Elecktrotech.,' Vol. 3, p. 67, 1914.

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in series with another condenser C_0. In case (1) C_0 is the contact capacity; in case (2) it is the capacity of the part of the air gap not occupied by the dielectric. We shall suppose the contact resistance R_0 to be infinite. This seems reasonable in the case of the air gap, and in the case of mercury electrodes absorption effects are usually enormously greater than "true leakage," so that the supposition is probably justifiable. Let us consider the capacity at frequency $\omega/2\pi$. We have

$$\text{Capacity} = C + \Delta C = C + \frac{kC}{1 + \omega^2 T^2},$$

where

$$k = \frac{C_0}{C_1}, \quad T = R_1 (C_1 + C_0), \quad \text{and} \quad C = \frac{C_1 C_0}{C_1 + C_0}.$$

Therefore

$$\text{effective capacity} = \frac{C_1 C_0}{C_1 + C_0}\left[1 + \frac{C_0}{C_1} \cdot \frac{1}{1 + \omega^2 R_1^2 (C_1 + C_0)^2}\right]. \quad \ldots \quad (7)$$

This represents the observed capacity of the sample in terms of the true constants of the material. Now take the case of the air-gap method. Denote the capacity as measured by this method by C_g. In using the simple formula for this method we put

$$\text{Observed capacity} = \frac{C_g C_0}{C_g + C_0}.$$

Equating this with the above expression we find

$$\frac{C_g}{C_g + C_0} = \frac{C_1}{C_1 + C_0}\left[1 + \frac{C_0}{C_1} \cdot \frac{1}{1 + \omega^2 R_1^2 (C_1 + C_0)^2}\right]$$

$$= \frac{C_1}{C_1 + C_0} \quad [1 + \alpha] \quad \text{say.} \quad \ldots \quad (8)$$

From this it is obvious that the calculated value C_g is higher than the true capacity of the sample, C_1, the difference depending on the factor α. Equation (8) may be written

$$C_g = \frac{C_1 (1 + \alpha)}{1 - \frac{C_1 \alpha}{C_0}} = \frac{C_1\left[1 + \frac{C_0}{C_1} \cdot \frac{1}{1 + \omega^2 R_1^2 (C_1 + C_0)^2}\right]}{1 - \frac{1}{1 + \omega^2 R_1^2 (C_1 + C_0)^2}}.$$

Put $\frac{C_0}{C_1} = z$. Then

$$C_g = \frac{C_1\left[1 + \frac{z}{1 + (1 + z)^2 \cot^2 \delta}\right]}{1 - \frac{1}{1 + (1 + z)^2 \cot^2 \delta}},$$

where δ is the true loss angle of the material, *i.e.*, $\cot \delta = R_1 C_1 \omega$.

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This reduces to

$$C_g = C_1\left[1 + \frac{\tan^2\delta}{1+z}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (9)$$

Thus the observed capacity is larger than the true capacity by an amount which depends on $z = C_0/C_1$ and tan δ. The larger the value of z—*i.e.*, the more nearly the sample fills the air gap—the nearer does the observed value C_g approximate to the true value C_1. If the sample is a poor insulator, then tan δ may be quite large. For a good conductor tan δ approaches an infinite value, and thus the method would give an infinite value for the capacity of such a material, although the true capacity might be quite small.

Now take the case of mercury electrodes, or rather the case in which metal electrodes of any sort are made to give as good a contact as possible with the dielectric, and the contact is then assumed to be perfect. Again the capacity is given by (7), *i.e.*, if C_m is the capacity given by this method

$$C_m = \frac{C_1C_0}{C_1+C_0}\left[1 + \frac{C_0}{C_1}\cdot\frac{1}{1+\omega^2R_1^2(C_1+C_0)^2}\right]$$

$$= C_1\frac{\left[1+\dfrac{z}{1+(1+z)^2\cot^2\delta}\right]}{1+\dfrac{1}{z}} \quad \ldots\ldots\ldots\ldots \quad (10)$$

where $z = C_0/C_1$ as before. In this case z is known to be very large, *e.g.*, if the dielectric were 1 mm. thick and there was an air film between electrode and dielectric equivalent to 0·001 mm. of dielectric, then we should have $z = 1{,}000$. Thus, as an approximation we may write

$$C_m = C_1\left[1-\frac{1}{z}\right]\left[1+\frac{z}{1+z^2\cot^2\delta}\right] \quad \ldots\ldots\ldots \quad (11)$$

Using the expressions (9) and (11), we may calculate the amounts by which the observed capacity, as measured by these two methods, will differ from the true capacity C_1. The results of such a calculation are given in the Table I. The values taken for z are 9 for the air-gap method and 1,000 for the mercury electrode method.

TABLE I.—Ratio of Observed Capacity to True Capacity.

δ.	tan δ.	$\frac{C_m}{C_1}$.	$\frac{C_g}{C_1}$.
° ′			
0 35	0·01	1−0·001	1·00001
5 43	0·1	1−0·001	1·001
17 33	0·316	1−0·0009	1·01
45 0	1·0	1·000	1·1
84 18	10·0	1·099	11·0
89 26	100·0	10·9	1001·0

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We see that in every case the air-gap method gives the higher value of capacity. Thus high values of capacity obtained by the mercury method cannot be explained in this way. In the case of a good conductor the value of cot δ approaches zero, and thus the capacity observed by the mercury method approaches the value

$$C_1\left[1+\frac{1}{z}\right]^{-1}[1+z],$$

i.e., C_0, the contact capacity.

Consider now the power factor measurements. The quantity measured by a bridge method is the effective conductance, *i.e.*, $\omega C_e \tan \delta_e$, where C_e is the effective capacity and $\tan \delta_e$ the effective value of the tangent of the loss angle. The value of the effective conductance in terms of what we have considered to be the true constants of the material is given by (5). Thus for the mercury method in the present case we have

$$C_m \tan \delta_m = C\,\frac{\omega kT}{1+\omega^2T^2} = \frac{C_1C_0}{C_1+C_0}\cdot\frac{\omega\cdot\frac{C_0}{C_1}R_1(C_1+C_0)}{1+\omega^2R_1^2(C_1+C_0)^2}$$
$$= \frac{C_1}{1+\frac{1}{z}}\cdot\frac{z(1+z)\cot\delta}{1+(1+z)^2\cot^2\delta}.$$

Since z is large compared with unity we may write this

$$C_m \tan \delta_m = \frac{C_1}{1+\frac{1}{z}}\cdot\frac{\tan\delta}{1+\left(\frac{\tan\delta}{z}\right)^2},$$

and by reference to (10) we see that this becomes

$$\tan\delta_m = \tan\delta\left[1+\left(\frac{\tan\delta}{z}\right)^2\right]^{-1}\left[1+\frac{z}{1+(1+z)^2\cot^2\delta}\right]^{-1}. \quad . \quad . \quad (12)$$

In the case of the air gap the test condenser is considered as consisting of two condensers in series, the sample, and the rest of the air gap. Thus the loss angle is calculated from the usual formula for condensers in series, which in the case considered becomes

$$\frac{C_0\tan\delta_g}{C_0+C_g} = \frac{C_0\tan\delta}{C_1+C_0},$$

$\tan\delta_g$ being the value obtained for $\tan\delta$ when using the air-gap method. Thus

$$\tan\delta_g = \tan\delta.\frac{C_0+C_g}{C_0+C_1}.$$

Substituting from (9) this becomes

$$\tan\delta_g = \tan\delta_1\left[1+\frac{\tan^2\delta}{(1+z)^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

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We may now use (12) and (13) to calculate the errors in tan δ which may be expected in the different cases, exactly as was done for the capacity values. Table II gives the values for the cases (*a*) air gap where $z = 9$, (*b*) mercury electrodes where $z = 1{,}000$.

TABLE II.—Ratio of Observed and True Values of tan δ.

δ.	tan δ.	$\frac{\tan \delta_m}{\tan \delta}$.	$\frac{\tan \delta_\eta}{\tan \delta}$.
° ′			
0 35	0·01	$1-10^{-7}$	$1+10^{-6}$
5 43	0·1	$1-10^{-5}$	$1+10^{-4}$
45 0	1·0	0·999	1·01
84 18	10·0	0·909	2·0
89 26	100·0	0·090	101·0

Thus the errors only become considerable when the loss angle is very large, and in this case the result obtained with mercury electrodes is the more accurate. With very large power factors, the air-gap method will give too high a value and the mercury electrode method too small a value. In practice the air-gap method would probably be unworkable with materials of high power factor, owing to want of uniformity of thickness and of perfect flatness in the specimen.*

BARLOW† maintains that owing to the high contact resistance it is useless to employ mercury electrodes in order to measure the true volume resistivity of dielectrics. This is probably true in the case of direct current measurements, but the calculations here given show that when the current is alternating the observed value for the loss angle approximates closely to the true value, even when the contact resistance is infinitely great. In fact, if it is desired to measure the true resistivity of the material, measurements should be made at radio frequencies, for the effective conductivity at frequency $\omega/2\pi$ is given by

$$\sigma = \frac{Ck\omega^2 T}{1 + \omega^2 T^2}.$$

When the frequency is very high, so that $\omega^2 T^2$ is large compared with unity, this becomes

$$\sigma = \frac{Ck}{T} = \frac{C_1 C_0}{C_1 + C_0} \cdot \frac{C_0}{C_1} \cdot \frac{1}{R_1 (C_1 + C_0)}$$
$$= \frac{1}{R_1} \cdot \frac{1}{\left(1 + \frac{1}{z}\right)^2}.$$

* HAWORTH, 'Proc. Roy. Soc.,' A, Vol. 81, p. 221, 1908.

† BARLOW, 'J. Inst. Elec. Eng.,' Vol. 62, p. 711, 1924.

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And since z is large, this approximates to the true conductivity $1/R_1$. Consider BAIRSTO'S* results for slate and marble. He found that as the frequency is raised the effective conductivity rises, rapidly at first, but afterwards more and more slowly, the value becoming approximately constant at the highest frequencies used (of the order 10^6 cycles per second). This value probably corresponds to the true resistivity of the material.

VIII.—*Conclusion.*

It is not suggested that all the observed phenomena of absorption, dielectric hysteresis and power loss are due to these contact effects. In the case of substances which are not homogeneous in structure MAXWELL'S theory applies, and "after-effect" phenomena would be observed whether these surface effects existed or not. In the case of quartz, iceland spar, etc., it is difficult to believe that heterogeneous structure provides an adequate explanation of the observed phenomena, and it is here that the present theory provides a more probable explanation. It is significant that there is a lack of agreement between the results of different observers on the decay of the absorption current with time. If this is largely a function of the electrode contact capacities and resistances, then the discrepancies are readily explained.

Another case in which this theory is of importance is that in which the dielectric is of comparatively poor quality, *e.g.*, celluloid, glass and porcelain (especially at high temperatures), porous dielectrics such as fabrics and paper, slate, and many other insulators of importance commercially. In such cases it is quite probable that the insulator may better be regarded as a solid electrolyte, in which the ionic mobility is small, than as a true dielectric, and the effect of heterogeneous structure may be small compared with the contact effect. Many of the curious phenomena observed in such cases are easily explained by this theory. For example, the large positive temperature coefficient of the effective capacity of porcelain† is a direct consequence of the negative temperature coefficient of its resistance, for it is obvious from equation (7) that the smaller the value of R_1 the higher is the effective capacity.

The theory also suggests a possible explanation of the difference between the properties of solid and liquid dielectrics. Liquids do not show residual charge phenomena,‡ and when they are subjected to an alternating voltage the power loss is practically independent of the frequency.§ This may be due to the fact that in the case of a liquid (which wets the electrodes) the electrical contact is much more perfect, so that the contact resistance and capacity are no longer of importance. The power loss is simply due to the resistance of the dielectric, and is thus independent of the frequency. In this case the effective capacity is independent of the resistivity of the material, and the temperature coefficient of capacity is negative, and not positive, as it is in the case where

* BAIRSTO, 'Proc. Roy. Soc.,' A, Vol. 96, p. 363, 1920.

† HAWORTH, 'Proc. Roy. Soc.,' A, Vol. 81, p. 221, 1908.

‡ VON SCHWEIDLER, 'Ann. der Phys.,' Vol. 24, p. 711, 1907.

§ PUNGS, 'Arch. Elektrotech,' Vol. 1, p. 329, 1912; BRYAN, 'Phys. Rev.,' Vol. 22, p. 399, 1923.

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contact effects are large. CASSIE* found that the temperature coefficient of capacity was positive for solids and negative for liquids.

A further interesting point which arises is the behaviour of the material under very high voltages. It seems probable that in this case the " contact capacity " would break down, so that the contact resistance would no longer be very great compared with the volume resistance. If this were so, then the " absorption current " would no longer be large compared with the leakage current. The behaviour of solid dielectrics under very high voltages has been studied by H. H. POOLE.† He found that whereas at low voltages measurements of leakage resistance are very difficult owing to the absorption current being much larger than the leakage current, at high voltages this is not the case, but the absorption effect, if it then exists, is completely masked by the leakage current.

Thus there is a considerable amount of experimental work for which the present theory provides an explanation. Quantitative comparisons are very difficult owing to the complexity of the results, but it is hoped in the near future to carry out an investigation on various dielectrics in order to see how far these considerations can be applied quantitatively.

* CASSIE, ' Phil. Trans. Roy. Soc.,' A, Vol. 181, p. 1, 1890.

† POOLE, ' Phil. Mag.,' Vol. 32, p. 112, 1916 ; Vol. 34, p. 195, 1917 ; Vol. 42, p. 488, 1921.

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XIII.

A MAGNETOMETER FOR THE MEASUREMENT OF THE EARTH'S VERTICAL MAGNETIC INTENSITY IN C.G.S. MEASURE

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ABSTRACT.

A comprehensive description of the theory, the practical construction, use and limits of error of a new form of magnetometer for the determination of the earth's vertical magnetic intensity, V. The method of direct annulment of the earth's vertical field is utilised. The principle of the magnetometer involves the use of a flat vibrating coil, and is carried out as follows : The Helmholtz coil system is set up with its axis truly vertical. A small, flat, very light coil is suspended about an axis of vibration at the centre of the Helmholtz system. The plane of the coil is vertical and its axis of vibration horizontal. A relatively large alternating current is sent through the vibrating coil, at the frequency of resonant vibration, resulting in a considerable vibration from the interaction between the alternating current and the earth's vertical magnetic field. The vibration is reduced to zero, when the applied vertical field exactly annuls the earth's vertical component, and the determination of V consists of observations of the current required to give an exact condition of rest or balance on the vibrating coil. The value of this current multiplied by the known constant of the Helmholtz coil gives the value of V in C.G.S. measure corresponding to that reading.

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I.—*Introduction.*

The measurement of the vertical component of the earth's magnetic field is a less simple operation than that of the horizontal component.

The horizontal field measurements are on a satisfactory basis, whether made by the swinging magnet method, or by the more recently developed electric magnetometers, in which known magnetic fields may be provided by means of known currents flowing through coils of known dimensions.

In the horizontal intensity magnetometers an indicating system is suspended or supported so that it can turn about a truly vertical axis. In a vertical intensity instrument, the corresponding suspension of an indicating device about a truly horizontal axis is by no means so simple. At present, therefore, vertical intensity measurements are deduced indirectly from horizontal intensity measurements combined with measurements of the angle of dip. This latter quantity cannot be measured with very great precision, and the accuracy of the finally deduced vertical intensity becomes less as the angle of dip becomes greater towards the magnetic poles.

The advantages of a direct method of measuring vertical intensity need no stressing; quite recently a novel absolute vertical intensity magnetometer has been developed by D. La Cour in Denmark.* In this instrument a large flat circular coil is turned about a horizontal axis from one position with its plane horizontal to the other similar position displaced 180° from the first. The flux linkage so produced is balanced by the flux linkage with the secondary winding of a mutual inductance when a known current change is made in the primary winding. From the published information on the instrument, the method appears to have considerable value and the accuracy attainable will doubtless be improved by further development.

The vertical intensity magnetometer forming the subject of the present paper makes use of the principle of balancing the vertical component of the earth's field by a vertical field produced by a steady current passing through a Helmholtz coil system. Some possibilities in this direction have been indicated by F. E. Smith† in his description of the

* D. La Cour, 'Terrestrial Magnetism,' Vol. 31, p. 153, 1926.

† F. E. Smith, 'N.P.L. Collected Researches,' Vol. 18, Paper 15, p. 317 (or 'Phil. Trans. Roy. Soc.,' Vol. 223, p. 175, 1922).

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horizontal force magnetometer developed at the National Physical Laboratory, along the principle first described by Sir ARTHUR SCHUSTER.* This principle, applied to horizontal intensity measurements, was to orient the Helmholtz coil about a vertical axis and to observe the angle at which $H = F \cos \alpha$, where F is the actual field produced at the centre of the coil and α is the angle which its horizontal axis makes with the magnetic meridian.

This principle could have been applied to a vertical force magnetometer, but the difficulties of determining when the resultant small field is truly horizontal are great. The direct annulment of the earth's vertical field appears to be a more satisfactory principle.

The principle of the magnetometer is carried into effect by setting up the Helmholtz coil system with its axis truly vertical. The measurement consists in adjusting and measuring the current through the coil system at that exact value necessary to produce a field equal and opposite to the earth's vertical component.

One of the most important features of such a method is the means whereby the exact annulment is observed. There are a number of possibilities; for example :—

(*a*) A small exploring coil may be rotated about a horizontal axis which must be directed very precisely along the magnetic meridian.

The electromotive force induced in such a coil will become zero at every instant when the vertical component of the resultant magnetic field is zero, since it is then directed along the meridian and is parallel with the axis of rotation of the exploring coil. The condition of zero induced electromotive force may be observed either on a direct current galvanometer, by the help of a commutator on the coil spindle, or on a vibration galvanometer tuned to a frequency resonant to the rate of revolution of the coil.

(*b*) A small indicating magnet might be suspended about a horizontal axis with its magnetic axis horizontal as in a vertical force magnetograph. The difficulties of using such a system are great, and the uncertainties regarding the conditions of balance consequent upon a gravity torque cannot easily be reduced to small quantities.

(*c*) A flat coil can be pivoted or suspended so that its plane is vertical and so that it can deflect about a horizontal axis. If a steady current is sent through the coil, a deflection will result. The deflection will become zero when the vertical field is zero.

(*d*) A very light coil can be suspended so that it may vibrate about a horizontal axis in a similar manner to a coil-type vibration galvanometer. If such a coil is supplied with alternating current of the same frequency as its free resonant vibration, it is very sensitive to a magnetic field in a suitable direction.

* Sir ARTHUR SCHUSTER, 'Terrestrial Magnetism,' Vol. 19, p. 19, 1914.

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It is probable that any of these methods could be made satisfactory from the point of view of sensitivity, convenience and accuracy, and some of them have been tried.

The method actually developed finally has been that outlined under (*d*), using a vibrating coil.

The reasons for choosing this method are—(1) No revolving parts are required in the apparatus in the magnetic field provided by the Helmholtz system, and (2) no delicate pivoting or suspension of the detecting device is necessary. The control forces required to give the vibrating coil a convenient resonant vibration frequency are relatively large, so that its statical position is located with great certainty and constancy.

II.—*Principle of the Magnetometer.*

The principle of the magnetometer is therefore carried into effect as follows :—The Helmholtz coil system is set up with its axis truly vertical. At the centre of the system a small, flat and very light coil is suspended about an axis of vibration. The plane of the coil is vertical, and its axis of vibration is horizontal. The coil is mounted in a frame in such a manner that these two conditions may be very nearly satisfied.

A relatively large alternating current at the frequency of resonant vibration is sent through the vibrating coil. Under these conditions a large vibration results from the interaction between the alternating current and the earth's vertical magnetic field. This vibration is reduced to zero when the applied vertical field exactly annuls the earth's vertical component. The determination of V consists in observing the current required to give an exact condition of rest or balance on the vibrating coil. This current, multiplied by the calculated or otherwise determined constant of the Helmholtz coil, gives the value of V in C.G.S. measure corresponding to that reading.

III.—*Simple Theory of the Vibration Coil Detector.*

The vibratory coil cannot be assumed to occupy the ideal position, in which its plane is truly vertical and its axis of vibration truly horizontal.

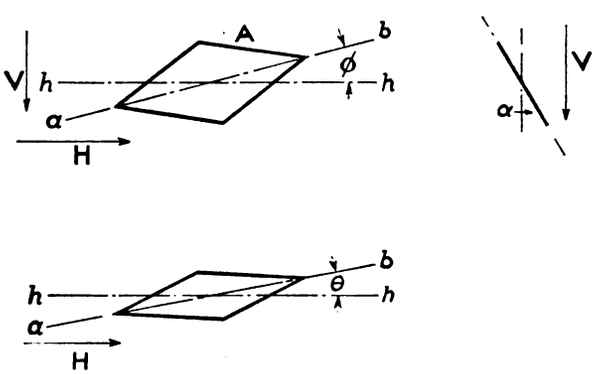

Fig 1.

Let the vibrating coil A (Fig. 1) be suspended about an axis of vibration *ab* and assume that the trace of this axis on the horizontal plane makes an angle θ with the magnetic

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meridian *hh*; the trace of the axis on the vertical plane makes a small angle ϕ with the horizontal plane. Finally, let the plane of the coil make a small angle α with the vertical plane containing the axis of vibration. It is convenient to consider the resolution of the vertical and horizontal components of the earth's magnetic field into components in the directions of the three principal axes of the vibrating coil. These axes are—

(1) perpendicular to the plane of the coil,

(2) along the axis of vibration of the coil,

(3) in the plane of the coil and perpendicular to its vibration axis.

Of these three components, only (3) is effective in producing a vibratory force on the coil when carrying the alternating current.

Resolving V and H in the plane of the coil and perpendicular to the axis of vibration, the resultant field which is effective in producing a vibrating force becomes

$$V \cos \phi \cos \alpha + H \sin \phi \cos \theta \cos \alpha + H \sin \theta \sin \alpha. \quad \ldots\ldots (1)$$

This is the field which is reduced to zero by the field applied by the current through the Helmholtz coil system.

The component of the applied vertical field U parallel to the resultant given by (1) is

$$U \cos \phi \cos \alpha, \quad \ldots\ldots\ldots\ldots (2)$$

where U is the vertical field produced by the current in the Helmholtz coil.

When a condition of balance is obtained, the earth's vertical component is given by equating (1) and (2), whence

$$V = U - H\left(\frac{\sin \phi \cos \theta \cos \alpha + \sin \theta \sin \alpha}{\cos \phi \cos \alpha}\right) \quad \ldots\ldots (3)$$

$$= U - H\left(\tan \phi \cos \theta + \frac{\sin \theta \tan \alpha}{\cos \phi}\right).$$

It is seen, therefore, that V cannot be determined from a single observation unless the small angles α and ϕ and the angle θ are known. It cannot be presumed that these angles may be known, but, fortunately, the correction term due to H can be entirely eliminated by orientating the vibrating coil and its frame about a vertical axis, through an angle of 180°. The effect is exactly equivalent to reversing the sign of H.

If we observe two values U_1 and U_2 of the applied vertical field required for balance in the two respective positions θ_1 and θ_2, where $\theta_2 = \theta_1 + \pi$, the mean value will give V exactly.

So that $V = \frac{1}{2}(U_1 + U_2)$, and

$$H(\tan \phi \cos \theta + \sin \theta \tan \alpha / \cos \phi) = \tfrac{1}{2}(U_1 - U_2).$$

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By means of levelling screws on the frame carrying the vibrating coil, an adjustment of the small angles ϕ and α can be made. By successive observations in the two positions giving U_1 and U_2 we can make $U_1 - U_2$ zero. The condition satisfied will then be

$$\tan \phi \cos \theta = - \sin \theta \tan \alpha / \cos \phi. \quad \ldots\ldots\ldots\ldots \quad (5)$$

It is seen, therefore, that a single adjustment of this kind cannot ensure ϕ and α being zero or of small importance for a small change in either of them.

It has been shown that the correction term exactly cancels out by orientation of the vibrating coil axis through 180°, but it is not desirable that accuracy of observation by the magnetometer should depend upon an exact orientation of the vibrating coil axis.

In practice ϕ and α may be considered always small, but θ may have any value we choose. An examination of the correction term due to H as given by (4) shows that, when θ is either small or nearly equal to π, *i.e.*, when the axis of vibration lies nearly along the meridian, the vibratory force due to H is mainly dependent upon the angle ϕ, which is the angle of inclination of the vibration axis to the horizontal plane. It is hardly dependent at all upon the angle α (between the plane of the vibrating coil and the vertical plane). Accordingly, when the vibration axis is nearly parallel to H, an adjustment of ϕ until $U_1 = U_2$ must result in causing ϕ to be very small indeed.

Again, suppose two values U_3 and U_4 are observed, with the axis of the vibrating coil in the two positions at right angles to the magnetic meridian ($\theta = 1/2\pi$ and $3/2\pi$), the vibrating force due to H will exist mainly on account of the small departure from verticality of the plane of the vibrating coil and hardly at all on account of the inclination of its vibration axis. By successive observations of U_3 and U_4 and adjustment of the angle α by means of the levelling screws on the coil mounting, the angle α can be reduced to a second order quantity. One or two alternate adjustments of ϕ and α by successive observations of U_1, U_2 and U_3, U_4 will result in the reduction of ϕ and α to negligibly small angles. When this adjustment has been obtained, the balance is undisturbed for all values of θ. The vibration axis can be in any azimuth and the same value of U will give the balance.

In the actual use of the instrument the observations are taken with the axis of the vibrating coil along the meridian (θ small). In the nature of its construction the vibrating coil is liable to small variations in the angle α which its plane makes with the vertical, but the angle ϕ is not apt to change much, since the suspensions are under considerable tension and so define the axis of vibration with precision and constancy. When θ is nearly 0 or π, small variations in α do not affect the balance.

Errors of the Method.—When making the observations there will be a small uncertainty in the orientation of the vibration axis, so that if θ is 0 in one position it will be $\pi + \Delta\theta$ in the second position, instead of π. The mean of the two readings U_1 and U_2 will then give a value for V which is in error by the amount $H\alpha\Delta\theta$.

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The procedure in the adjustments outlined above will enable the angle α to be reduced to 1/1000 radian with certainty. In order that the error $H\alpha\Delta\theta/V$ may not be greater than one part in a hundred thousand it will only be necessary to set θ to an accuracy equivalent to V/100 H. Actually θ can be set to an accuracy at least as high as 1/1000 radian, so that except in regions near the magnetic equator, where V/H will be small, there will not be any appreciable error due to $\Delta\theta$.

Two other sources of error arise in the adjustment of the instrument, due (*a*) to errors in setting the axis of orientation truly vertical and (*b*) in setting the axis of the Helmholtz coil parallel to this axis. These errors are dealt with in Appendix I, from which it will be seen that a small angular error ψ between the axis of orientation and the vertical will introduce an error into the measurement of V which at its maximum is equal to $H\psi$. It is seen, therefore, that this error is in the first order of small quantities. The axis of orientation must therefore be set vertical with great precision. The instrument is provided with very sensitive levels, and the turntable, which virtually defines the vertical axis of orientation, revolves on a ball race of large diameter. A careful test showed that the departure from verticality of the axis was certainly not greater than 1 part in 50,000 and was probably less than half this amount.

IV.—*Description of the Magnetometer.*

(*a*) *Helmholtz Coil System.*—This system comprises the same marble cylinder which forms the essential part of the horizontal force magnetometer described very fully by F. E. Smith in the paper previously referred to. For full details of the measurements of the diameter and pitch of the windings, etc., the reader is referred to that paper. The variations of the radial and axial fields of the coil over a region near its centre have also been fully investigated and need not be discussed here. It will suffice to say that within a sphere of 2 cms. diameter the maximum difference in axial intensity from that at the centre of the system is less than 4 parts in 10^6.

The calculated constant of the coil system is

$$U_t = 3{\cdot}59595\, i[1 - 7{\cdot}9 \times 10^{-6}(t - 17)],$$

where U_t is the axial magnetic force in C.G.S. units when the cylinder is at the temperature $t°$ C., and i is the current in absolute measure.

The constant $+ 7{\cdot}9 \times 10^{-6}$ is the measured thermal expansion coefficient of the marble in length and diameter. The constant 3·59595 is believed to be accurate to 1 part in 10^5 from the measured dimensions. This value refers to the dimensions measured in 1921. These may have slightly altered since then, but the present observations are on the assumption of unchanged dimensions.

The original base and turntable have been used in the present magnetometer, but in order to support the cylinder with its axis vertical, a non-magnetic bronze platform has been made to fit, on its under-side, the existing cradles on the turntable, and to provide

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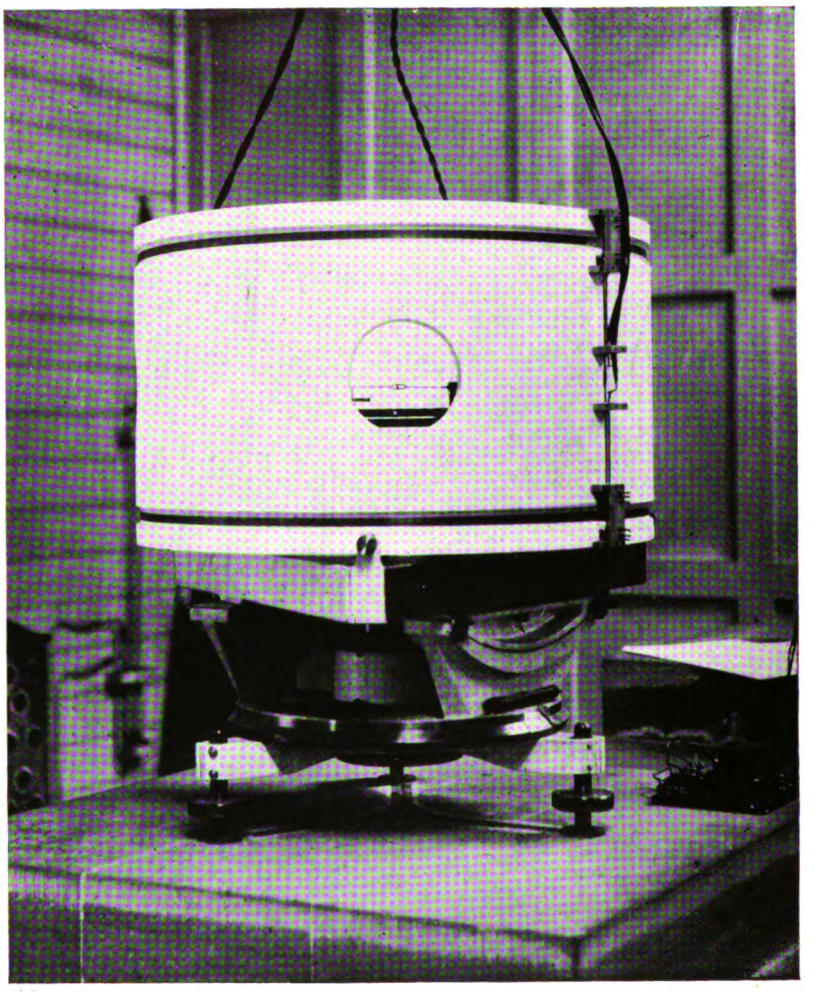

FIG. 2.

[*To face p.* 263.

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on its upper surface three levelling and three centring screws for the support and adjustment to verticality of the marble cylinder.

The general appearance of the magnetometer will be seen by reference to the photograph, Fig. 2, in which the triangular stiff platform can be seen. The turntable is provided with an accurately engraved angular scale, whereby it may be set in azimuth at any desired angle to an accuracy of a few seconds of arc. The levels on the base are very sensitive and enable the axis of the turntable to be set vertical to an accuracy of about 1 part in 100,000.

Within the cylinder is supported a circular bronze table upon a tubular pillar fixed to the turntable. Upon this table is mounted the vibration detector coil. This can be seen in the photograph, whereby its small size may be judged. The detecting coil and its mountings are shown in more detail in Fig. 3. They consist essentially of the very light, flat diamond-shaped coil C strained between phosphor-bronze suspensions, which are carried in a frame provided with means for adjusting the tension for tuning purposes.

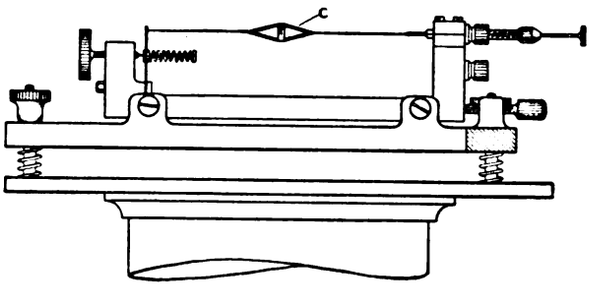

FIG. 3.

The coil in its frame is an independent unit, so that replacement is simple in case of accidental breakage of the suspensions. The frame and coil are mounted on a three-armed base, which registers on three bolts projecting up from the table. By means of bronze springs underneath and spherical seating nuts above, the adjustments to the vibration coil are made, whereby its axis of vibration is made horizontal and its plane is made vertical. The clamping arrangements also enable the coil to be adjusted so that its centre is approximately at the centre of the Helmholtz system. A sheet copper cover, with a hole facing the small mirror on the coil, serves to protect it from draughts and other disturbances, which otherwise cause the light spot to wander up and down on the scale.

There is considerable latitude in the design of the vibration coil, but that used in the experiments had 100 turns of enamelled wire of 0·05 mm. diameter. Its length was about 2 cms. and its width about 1 cm. The mass of the coil and mirror was about 150 mgms. The vibration frequency can be chosen to have any suitable value, between wide limits, by using the appropriate control torque given by the suspensions. Frequencies between about 100 and 5 cycles per second were tried. In all sizes of vibration coil used greater sensitivity was obtained the lower the resonant frequency chosen.

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But at the lowest frequencies the time taken by the vibration coil to come to rest was so long that the observations became very tedious. A frequency of 15 cycles per second was found to be a good compromise. The damping decrement of the vibration coil was such that with an initial vibration of the light spot of 1 cm. on the scale, a time of about 20 seconds was required for the spot to come sensibly to rest. The sensitivity is such that a vibration of about 25 mms. on a scale at 2 metres distance results from the application of a field of 40 γ. The condition of rest of the vibrating spot corresponds to a residual vibration less than 0·1 mm., since the eye is very susceptible to a vibration of about 15 per second.

The coil is small enough in size to ensure that the mean field of the Helmholtz system over the region occupied by it does not differ from that on the axis by more than 2 parts in 10^6.

(*b*) *Auxiliary Apparatus.*—The auxiliary apparatus necessary to the magnetometer, in addition to galvanometer lamps, scales and batteries, are—

(1) a small source of alternating current at a frequency of about 15 cycles per second,

(2) means of measuring and adjusting the Helmholtz coil current to a high degree of accuracy.

(1) Various sources of alternating current were tried, including a self-maintained tuning fork, a valve oscillator, and an alternator. Of these only the alternator proved satisfactory, since with the other sources, a perfect balance was unobtainable even when the magnetic field had been exactly compensated. A residual vibration of 15 or 20 mms. resulted when either a tuning fork or valve oscillator was used as source. This was puzzling at first, but was ultimately found to be due to heating effects of the current. The vibration coil is very small in surface, and, in order to obtain sensitivity, it is necessary to send a relatively large current through it (70 milliamperes). The energy dissipated in the coil is about 0·2 watt. Owing to an inevitable lack of symmetry of the coil, the alternate heating and cooling of each cycle sets up a small vibratory force, which, of course, is quite independent of the magnetic field in which the coil is supported. With a symmetrical wave form, however, the heating and cooling should have a frequency twice that of the source, and hence at the most should produce only a minute forced vibration at the double frequency, since the vibration coil is completely out of tune to this frequency. The case is, on the other hand, otherwise if harmonics of frequency $2n$ are present. An unsymmetrical wave form results, so that the heating during one half-wave is not equal to that during the next half-wave. A vibrative force at the resonant frequency will result and will cause a residual vibration which cannot be reduced to zero by any adjustment of the magnetic field. Various kinds of circuit were tried, but no absolutely perfect balance could be obtained with any other source than an alternator. With almost any kind of alternator, however, a perfect balance can be obtained.

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The alternator used was a small machine run as a rotary converter from a 6-volt battery. With the ball bearings and special brushes supplied the current taken was only 0·4 amp. A governor of the frictional gramophone type was provided to maintain the speed sufficiently constant for the alternator to remain sensibly synchronous with the frequency of the vibrating coil. At the normal speed the root mean square voltage was about 4, which provided 70 milliamperes in the vibrating coil. The speed continually varied slightly in use, but this did not matter in practice, since the condition of a slight want of balance on the vibration coil always showed itself when the observation was continued for one minute. The small alternator behaved very satisfactorily throughout the experiments. In general, it was only necessary to adjust the speed once or twice during a run of about an hour and a half. In some cases the whole run was made without any readjustment.

(2) In some cases the current measurement was made by a straightforward potentiometer method using a standard 1-ohm resistance coil. This method is very flexible, and by its means measurements may be made in any part of the world using the same arrangement. It requires an independent potentiometer, however, and necessitates repeated adjustment of the potentiometer current during a run. In the later runs the independent potentiometer was dispensed with and was replaced by a direct potentiometer method, whereby part of the potential difference on the resistance combination was balanced against a standard cell. The most convenient arrangement for this is as shown in Fig. 6 in Appendix II, in which the corrections and calibration of the current-measuring system are discussed.

In all cases where observations are made at a fixed station, the direct potentiometer method has much to recommend it. It can readily be adapted to give direct readings of the change in V over a range of a few parts in a thousand. On an instrument intended to be used at places widely separated, where V might have values ranging from 0 to 0·6 C.G.S. units, a separate potentiometer system would be much more flexible but not quite so accurate.

V.—*General Arrangement of the Apparatus.*

The general lay-out of the magnetometer and its auxiliaries is as shown in Fig. 4. The current-measuring system shown in the figure was that actually used, but the scheme suggested in Appendix II is preferable and will be used on future occasions.

The Helmholtz coil system CC is set up about the vertical axis *ab*. Its windings are supplied with current from a 20-volt battery V_1. Rough adjustment is provided by a three-dial rheostat P with good contacts and carrying capacity for the current of about 1·2 amp. The current-measuring arrangements actually used consisted of the standard resistance combination R_1, of about 0·8 ohm, in series with standard resistance R_2 of 0·01 ohm. R_2 was shunted by a slide wire S of about 0·76 ohm, in series with which was a diluting resistance S_1 of 1 ohm to increase the openness of the scale of S in terms of change of current. This combination enables a potential difference of 1·0183 volt to

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be obtained with potential contact at the middle of S when the appropriate current is passed through the Helmholtz coils and the resistance combination. The standard cell T, galvanometer and tapping key enabled the potential difference observed to be adjusted to exact equality and opposition to that of the standard cell.

From a knowledge of the exact values of R_1, R_2, S and S_1, the current corresponding to any reading on the slide wire was calculated. The rest of the current circuit consists

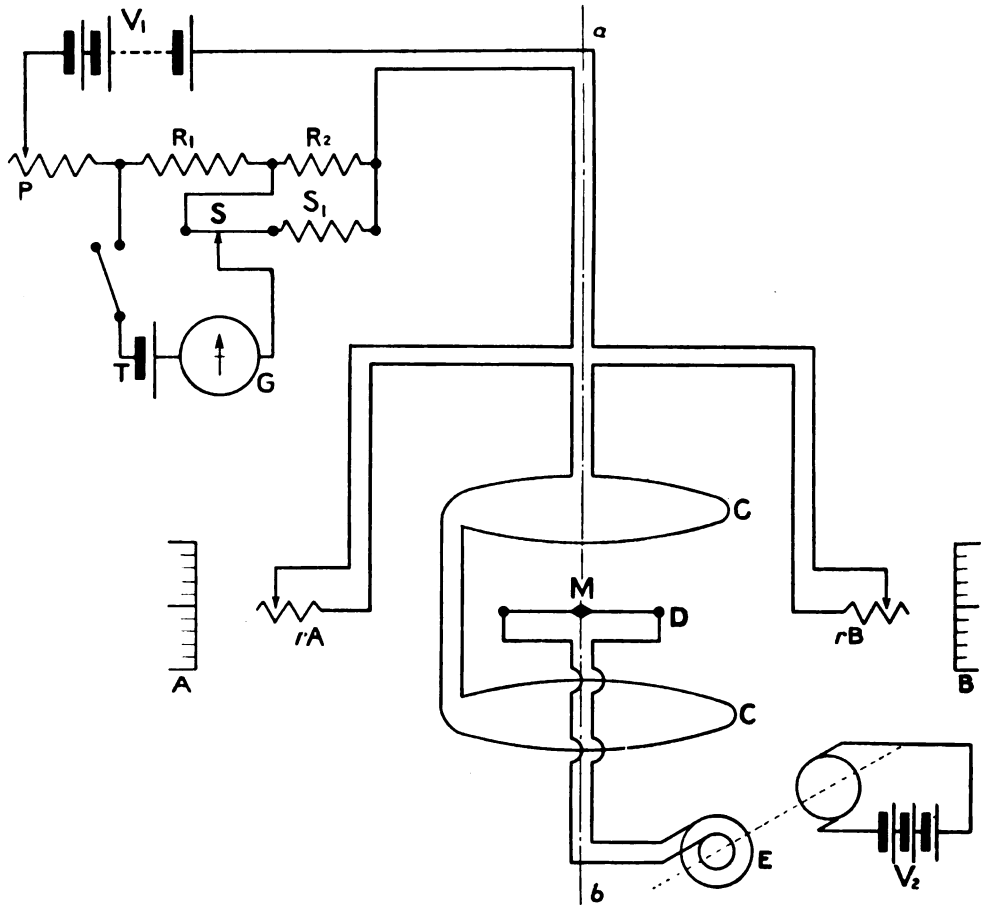

FIG. 4.

of the fine adjustment resistances rA, rB, each in the form of a mercury channel and short-circuiting slider. These resistances are placed respectively near the vertical scales A and B, upon which the vibration of the light spot reflected from the vibration coil is received.

The line joining AB is approximately at right angles to the meridian, so that the axis of the vibration coil D lies approximately along the meridian when the light spot is on A or B. The vibration coil is mounted as at D and carries a very small mirror M. The coil is connected to the slip rings of the little motor converter shown diagrammatically at E. The direct-current motor end is connected to a 6-volt battery V_2.

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In addition to the scales, A, B, and the galvanometer lamps near them (not shown), there are two other lamps and scales along a line perpendicular to the plane of the diagram through the centre of the system. These latter are used at the commencement of a series of observations for the purpose of adjusting the plane of the vibration coil to its electrically vertical position. The actual observations are taken only on scales A and B, alternately.

In the actual experiments made at the Laboratory, the direct current measuring apparatus and battery V_1 were, for convenience, in the main building, whilst the magnetometer itself was in a special hut removed from local magnetic disturbances arising within the Laboratory. The motor converter with its battery V_2 was set up just outside the hut at a distance of about 4 metres from the magnetometer. It was so arranged that the vibration of the light spot was visible through the doorway when adjusting the governor and the speed of the converter to that corresponding to resonance of the vibration coil.

A test was made to determine the stray field produced by the permanent magnets of the motor converter at the centre of the magnetometer. With the position of the converter such that the stray field was vertical, it was found to be equal to about $5\,\gamma$ at 4 metres distance. By turning the converter so that the stray field is horizontal, therefore, the error becomes quite negligible.

A vertical force magnetograph was constructed and set up for recording continuously the changes in V during a series of observations on the magnetometer. The instrument consisted of a cobalt-steel magnet 5 mms. diameter and 20 mms. long. The magnet was mounted transversely on a brass spindle provided with adjustable gravity control weights at right angles to one another and to the spindle. Thin phosphor-bronze suspensions, attached one to each end of the spindle and lightly strained in line to supports on a base, formed the horizontal axis of suspension.

A damping vane between copper plates served to render the system just aperiodic. The gravity control was adjusted to be nearly equal and opposite to that due to the vertical component acting on the magnet. The control force was finally adjusted by means of a small magnet which provided a weak horizontal field directed along the magnetograph magnet axis.

The image of a Nernst filament formed by reflection from a mirror on the magnet fell on a narrow horizontal slit, having first passed through a prism to turn the movement of the reflected ray into a horizontal plane. A long band of photographic paper slowly revolving behind the slit recorded the variations in V. The sensitivity was such that a displacement of about 1 mm. was produced by a field change of $1\,\gamma$. The photographic paper moved at the rate of about 40 cms. per hour. The effect of the magnetograph magnet upon the magnetometer was quite negligible at the distance of 4 metres which separated them.

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VI.—*Experimental Observations with the Magnetometer.*

(*a*) *Adjustments.*—A few preliminary runs having been made in order to gain experience in the use of the instrument, it was set up in the magnetic hut in the Laboratory grounds. It was soon seen, both from the magnetometer and from the magnetograph record, that accurate observations were impossible during the daytime owing to train and tram disturbances; continuous, rapid, and very large excursions (up to 100 γ in magnitude) occurred all the time. The accurate runs were therefore made at night during the hours between 1.30 and 4.0 a.m.

The turntable was first levelled as accurately as possible, using the sensitive levels on it. The resulting position of the axis of turning was vertical to a probable accuracy of 1/50,000 radian. The marble cylinder was then centred and levelled on its platform by the aid of a dial gauge capable of indicating an eccentricity of 0·01 mm. The gauge was held in a rigid clamp so that its foot pressed against the outer cylindrical surface of the cylinder. By alternate observations at the lower end of the cylinder and adjustment of the centring screws, followed by observations at the upper end and adjustment of the levelling screws, the cylinder was brought into a truly concentric position with respect to the axis of turning.

The galvanometer lamps and scales were then adjusted so that the image of the cross-wires on the lamp lens, formed after reflection from the small mirror on the vibration coil (at rest), fell on definite lines on the scales, thus locating the two positions of the vibration coil axis approximately along the magnetic meridian. The other two lamps and scales were set up and adjusted to correspond to positions of the axis of D at right angles to the meridian. The current having been switched on and having become steady, the next adjustments made were those of the vibration coil. The light spot was brought to position on scale A, and the mercury rheostat rA was adjusted until balance was obtained. The magnetometer was then oriented until the light spot was on scale B. A small vibration of the light spot resulted. By temporary adjustment of rB for balance, an estimate of the difference in current for balance in the A and B positions was obtained. The rheostat rB was then placed midway between its new and old positions. The vibration, now reduced to about half its original value, was reduced to zero by adjustment of the axis of vibration of the coil by means of the appropriate levelling screw on D. A similar procedure was carried through with the vibration coil axis in the two E.W. directions, except that adjustment to equality was now made by adjustment of the plane of the vibrating coil.

After one or two repetitions of each of the above adjustments, a condition was arrived at such that for any of the four positions of orientation of the vibrating coil axis the current for balance was the same to within 2 or 3 parts in 10,000.

These adjustments satisfactorily completed, the actual observations constituting a run were made as follows:—

(*b*) *Observations.*—(i) A calibration of the magnetograph was made by means of a known small current passed through a two-turn coil in a horizontal plane centrally

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disposed around the small suspended magnet constituting the essential part of the magnetograph. This current was such that a change of 25 γ was made in the vertical field. The sensitivity of the magnetograph was such that a deflection of its recording light spot of about 30 mms. resulted. The calibrating field was applied in both positive and negative directions.

(ii) The magnetometer was brought into the A position and balance of the light spot obtained by adjustment of rA. The light spot had crossed lines upon it and was observed by means of a lens. It was found possible to obtain a balance of great sharpness. The eye is peculiarly susceptible to the vibration at 15 cycles per second, and the condition of rest, when obtained, corresponded to a vibration less than 0·1 mm. in amplitude. Under the steady conditions appertaining to the night observations and on a magnetically quiet occasion the balance remained sensibly perfect for a minute or more at a time.

(iii) When the balance had been obtained, and was seen to be real and steady, a signal was given by means of a tapping key and lamp to an observer stationed at the current-measuring apparatus. He forthwith adjusted the slider on S until a balance on galvanometer G was obtained. The reading on S was noted, and the time at which the signalling flash occurred was also entered.

Simultaneously with the flashing signal to the observer a locating signal was given to the recording magnetograph. This took the simple form of lighting a lamp, situated about 10 feet in front of the narrow slit in the box containing the photographic paper upon which the magnetograph record was in course of exposure. This signal gave a fine line on the record and identified on it the instant at which the magnetometer observation was made.

(iv) The magnetometer was turned to the B position and the procedure outlined under (ii) and (iii) was carried out, the adjustment of current being made by means of rB.

A complete series of observations was thus made alternately on scales A and B throughout a period of about one and a half hours. During such a period between 30 and 40 observations were made.

At a few intervals, readings of the temperature of the marble cylinder of the magnetometer and of the standard resistance coils, R_1, R_2, were made. The standard cell was in an oil bath in a vault and its temperature remained absolutely steady throughout a run.

During the run observations were made to determine whether the rotary converter was remaining in tune with the vibrating coil. In order to ascertain this, one of the rheostats rA or rB was offset so as to produce a change in current sufficient to give several centimetres vibration of the light spot, and it was observed whether this vibration was as large as it should be; if it was too small a readjustment of speed was made. In general the small unavoidable variations of speed were such that the vibrating coil continually went into and out of tune, so that the vibration varied from a maximum to about one-half of this value. During any period of one minute, however, the light

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spot nearly always reached an amplitude nearly equal to its maximum amplitude. It was only occasionally, therefore, that an observation was in error owing to insensitivity resulting from the vibrating coil being out of tune.

In the course of every observation a small displacement equivalent to about 1 part in 10^4 in current was made, and a reading was not accepted unless the resulting small vibration was visible. At the conclusion of a series of observations, a second calibration of the magnetograph was made.

The observations, when obtained, were dealt with as follows :—From a knowledge of R_1, R_2, S and S_1, a factor was determined for converting differences from an arbitrarily chosen reading on S into differences in γ units from the base line value corresponding to the arbitrary reading. These γ differences were entered opposite the corresponding S readings for the alternate A and B observations numbered in sequence, 1, 2, 3, etc. The values for the mean of each pair of observations, 1 and 2, 2 and 3, 3 and 4, etc., were then taken. These are the final V differences, corresponding to the mean value of B at the instants 1 and 2, 2 and 3, etc.

The displacements from an arbitrary base line on the magnetograph record were next read from the record at the instants 1, 2, 3, etc. These, converted to γ differences, were then treated similarly to the magnetometer readings so as to give the mean values of the readings 1 and 2, 2 and 3, etc.

Finally, the differences between the magnetograph and the magnetometer deduced values of "γ difference" were taken. These latter should, of course, under ideal conditions, be quite constant. The variations in these differences give a measure of the uncertainty in any reading resulting from all the uncertainties in the chain of observations connecting them.

The final results deduced were the differences between the mean value of the differences of all the comparison readings and the individual difference readings. The mean variation from the mean given at the bottom of the last column is a measure of the probable amount by which any observation is different from the mean of a number of such observations. This latter figure is therefore a rough measure of the probable inaccuracy associated with the process of observation. It must not be confused, of course, with inaccuracies in the absolute measurement of the whole vertical field associated with constant errors in the current measurement and in the constants and the set-up of the magnetometer. The results of a typical series of observations just as taken are given in the table following.

Columns (1) and (2) give respectively the number and instant of the observation.

Columns (3) and (4) give the slide-wire readings for the A and B positions alternately, taken from an arbitrary zero position near the centre of its scale.

Column (5) gives the means between the successive pairs, and column (6) gives the corresponding values in γ units.

Column (7) gives the deduced mean values of γ differences, from an arbitrary base line, as read off from the magnetograph record.

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TABLE.

(1) Obs.	(2) G.M.T.	(3) Slide wire. A.	(4) Slide wire. B.	(5) Means, successive pairs.	(6) γ units.	(7) Magnetograph means, γ.	(8) Differences (6) − (7).	(9) Differences from mean of (8), γ units.
1	1–29–15	−0·025		−0·037	10·7	−13·8	24·5	−2·7
2	1–31–20		−0·050	−0·041	11·9	−14·2	26·1	−1·1
3	33–45	−0·032		−0·040	11·6	−14·5	26·1	−1·1
4	26–50		−0·048	−0·043	12·5	−14·7	27·2	0
5	41–15	−0·038		−0·043	12·5	−15·1	27·6	+0·4
6	42–40		−0·049	−0·042	12·2	−15·2	27·4	+0·2
7	48–20	−0·035		−0·043	12·5	−15·1	27·6	+0·4
8	50–38		−0·052	−0·046	13·3	−15·2	28·5	+1·3
9	53–32	−0·041		−0·046	13·3	−15·2	28·5	+1·3
10	55–01		−0·051	−0·041	11·9	−15·3	27·2	0
11	57–28	−0·031		−0·043	12·5	−15·5	28·0	+0·8
12	59–43		−0·055	−0·042	12·2	−15·6	27·8	+0·6
13	2–01–35	−0·030		−0·039	11·3	−15·8	27·1	−0·1
14	05–35		−0·048	−0·041	11·9	−16·0	27·9	+0·7
15	07–43	−0·034		−0·042	12·2	−16·5	28·7	+1·5
16	09–20		−0·051	−0·042	12·2	−16·3	28·5	+1·0
17	14–08	−0·042		−0·044	12·7	−15·9	28·6	+1·4
18	15–30		−0·046	−0·039	11·3	−15·7	27·0	−0·2
19	17–35	−0·032		−0·039	11·3	−15·7	27·0	−0·2
20	19–42		−0·045	−0·039	11·3	−15·7	27·0	−0·2
21	22–32	−0·031		−0·038	11·0	−15·9	26·9	−0·3
22	23–46		−0·045	−0·038	11·0	−16·0	27·0	−0·2
23	26–00	−0·026		−0·036	10·4	−16·4	26·8	−0·4
24	27–15		−0·035	−0·032	9·3	−16·4	25·7	−1·5
25	29–43	−0·035		−0·035	10·1	−16·3	26·4	−0·8
26	31–05		−0·040	−0·037	10·7	−16·3	27·0	−0·2
27	32–22	−0·025		−0·033	9·6	−16·1	25·7	−1·5
28	34–13		−0·038	−0·032	9·3	−16·0	25·3	−1·9
29	36–40	−0·032		−0·035	10·1	−16·5	26·6	−0·6
30	37–53		−0·056	−0·044	12·7	−16·9	29·6	+2·4
31	39–32	−0·016		−0·036	10·4	−17·3	27·7	+0·5
						Mean	27·2	
						Mean variation from mean 0·85 γ		

Column (8) gives the differences between corresponding readings in columns (6) and (7) in γ units.

Column (9) gives the differences between the mean of all the values in column (8) and each individual reading. The mean value of the differences shown in column (9) is 0·85 γ, and the maximum positive and negative values are 2·4 γ and 2·7 γ respectively.

The values of the various items by which the base-line value is calculated corresponding to the zero on the slide were

$R_1 = 1{\cdot}00036$ internat. ohm.

R_2 + leads = 5·5071 internat. ohm.

$x = 0{\cdot}37$ ohm.

$S + S_1 = 1{\cdot}76$ ohm.

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Voltage of standard cell at 15° C. = 1·01842 internat. volt.

Helmholtz coil constant at 16° C. = 3·59598.

Calculated base-line value $V_0 = 4315_2\ \gamma$ units.

In the present investigation, however, no particular importance is to be attached to this base-line value, since the aim has been rather to prove the system than to obtain results by the magnetometer.

It would seem, therefore, that, *as far as observational accuracy is concerned*, any single observation has a probable error not greater than 1 γ, and the maximum probable error is about 2·5 γ. From the same standpoint the probable error in base-line value as given by the mean of 30 observations would be about 0·3 γ, or less than one part in a hundred thousand in V, in England.

These conclusions are borne out by a number of runs made during a period of about three weeks. On many of these occasions the temperature of the magnetograph was not constant, and it became necessary to apply an estimated correction for the consequent drift in the magnetograph trace. The mean of the various values of " mean variation from the mean " was about 0·8 γ, and the maximum value was 1·2 γ.

Under steady magnetic conditions, therefore, it may be considered experimentally proved that the mean base-line value given by 30 observations taken during an hour would not be uncertain, due to observational errors, by more than 0·3 γ.

The variations of temperature and the uncertainties associated with the magnetograph from one day to another were such that it was not possible to test the constancy of base-line values given by the magnetometer.

The instrument has been transferred to the Abinger Magnetic Observatory, where a series of observations of V is in progress with a view to proving the accuracy of determination of a base line. It is hoped to publish the results of this further work at a later date.

VII.—*Probable Errors.*

The errors associated with a determination of V may be classified as due to (*a*) the method, (*b*) the measurements, and (*c*) the magnetometer. The errors associated with these factors are as follows :—

(*a*) *The Method.*—The vibrating coil may be slightly magnetic, but its mass is so small that the change in the resultant field due to hysteresis, etc., is probably quite negligible. A residual vibratory torque due to the heating and cooling of the coil and its suspensions may still be present even with the symmetrical wave form given by the alternator. If this torque were in phase with the main torque, a balance would still be obtainable by the appropriate resultant very small vertical field which would automatically be applied to obtain it. A reversal of the connexions to the vibrating coil would reverse the sign of the main torque but would not reverse a torque due to heating. A different current would then be needed for balance. An experiment showed no measurable difference due to reversal of the vibrating coil connexions. It is, moreover, exceedingly unlikely

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that a vibrating torque due to heating and cooling would not have a compoent in quadrature with the main torque. In such a case a perfect balance would be unobtainable by simple adjustment of the Helmholtz coil current. Actually the balance when obtained was very sharp.

A further experiment was tried in order to see the effect of an asymmetrical wave form. This consisted in superposing a small steady current upon the alternating current flowing in the detector coil. The effect of this was apparent, as no perfect balance could be obtained, thus proving that the thermally produced fundamental frequency torque had a component in quadrature with the main torque. The effect was small and the setting of the Helmholtz current for approximate balance appeared to be unaffected by reversal of the connexions on the vibrating coil.

The errors associated with the location of the vibrating coil have been discussed, and it has been shown that such errors are proportional to H, and therefore increase towards the magnetic equator. By the method of observing in two positions of the vibrating coil axis, 180° apart, it has been shown that the error may be reduced to less than 1 γ when H = 0·5 C.G.S. unit.

There is a possible source of small error in the inevitable lack of perfect geometrical symmetry of the vibrating coil. If the suspensions are not strictly coincident with the axis of vibration, a small torque will result, since they are considerably displaced from the centre of the Helmholtz coil and are therefore not in a zero vertical magnetic field when this condition is obtained at the centre. Such error must be very small, since there are about 400 cms. of wire in the coil at an effective radius of about 3 mms. from the axis, whereas the suspension will almost certainly be within 0·3 mm. of the effective axis of vibration and has an effective length of about 3 cms. only.

(*b*) The errors associated with the measurements are those due to inaccuracy of observation and uncertainty in measurement of the current in the Helmholtz coil. The sensitivity is such that variations in setting the current may amount to the equivalent of 2·5 γ in a single observation, with a probable variation of 1 γ. The mean of a series of 20 observations would have a probable error of less than 0·5 γ, due to limitation of sensitivity on the vibration coil.

The error associated with observing the current will include a small drift which may occur between the instant of the magnetometer observation and the observation of current on the potentiometer. This error is likely to be less than one part in a hundred thousand. The error in the finally deduced current might be as great as 1 part in 10^5, but the possible error in the unit in which the current is measured is 2 or 3 parts in 10^5, whether this unit is derived from standard resistances and a standard cell, or, as at the Laboratory, from an absolute unit given by the current balance.

(*c*) The errors associated with the magnetometer are due to errors in setting up and errors in its constant.

The former error has been discussed and was seen to be probably not greater than $H/10^5V$, or in γ units $H/10^5$.

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The error associated with the dimensions of the coil can only be estimated by a repetition of the measurements of length and diameter. Previous experience has shown that slow changes in the dimensions of marble cylinders do occur in the course of a few years, and they may in the present case amount to a few parts in a hundred thousand.

From these considerations it would appear that the complete magnetometer and its accessories form a means of obtaining a base-line value for vertical intensity which would be consistent to 1 or 2 γ, but that the absolute value of the field in C.G.S. units is subject to an uncertainty of about twice this amount.

In conclusion, my thanks are due to the Metrology Department for the design and construction of parts of the mounting, and to Mr. J. E. P. L. VIGOREUX, M.Sc., for much careful assistance in the observations and in the method of presenting the matter in Appendix I. My thanks are also due to the Director and to the late Dr. CHREE, F.R.S., for kindly interest in the development of the magnetometer.

APPENDIX I.

Discussion of Errors.

General case in which a Helmholtz coil system is set up with its axis nearly vertical and can be oriented about another axis nearly vertical. This latter is an axis of reference.

The detecting coil is set up with its plane nearly vertical and its axis of vibration pointing in any direction in a nearly horizontal plane; it is capable of orientation about the axis of reference.

Let OX, OY and OZ be three axes of reference such that OY is coincident with the nearly vertical axis around which the Helmholtz coil and the vibrating coil can be oriented.

Let U_M be the field produced by the Helmholtz coil and let its direction cosines be ξ, η and ζ, where ξ and ζ are nearly zero and η nearly unity.

Let the direction cosines of the vertical be λ, μ and ν, where λ and ν are nearly zero and μ is nearly unity.

Let the earth's field be F and its direction cosines be l, m and n. None of these is necessarily small.

Let the direction cosines of a line in the plane of the vibrating coil and perpendicular to its axis of vibration be α, β and γ, where α and γ are small and β is nearly unity.

The angle between F and the line α, β, γ is $\cos^{-1}(\alpha\xi + \beta\eta + \gamma\zeta)$.

The vibratory torque due to F is proportional to

$$F(\alpha l + \beta m + \gamma n).$$

Again, the angle between V_m and the line α, β, γ is

$$\cos^{-1}(\alpha\xi + \beta\eta + \gamma\zeta),$$

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and the vibratory torque is proportional to

$$U_M(\alpha\xi + \beta\eta + \gamma\zeta).$$

By adjustment of U_M to the value U_{M1} we can reduce the vibratory torque to zero, so that

$$F(\alpha l + \beta m + \gamma n) = U_{M1}(\alpha\xi + \beta\eta + \gamma\zeta). \quad \ldots\ldots\ldots \quad (1)$$

Now let the Helmholtz coil and the vibrating coil be oriented about the nearly vertical axis of reference OY through an angle π.

The new direction cosines become

$-\xi$	η	$-\zeta$
λ	μ	ν
l	m	n
$-\alpha$	β	$-\gamma$

By adjustment of U_M to the value U_{M2}, such that the vibratory torque is again zero, we have

$$F(-\alpha l + \beta m - \gamma n) = U_{M2}(\alpha\xi + \beta\eta + \gamma\zeta). \quad \ldots\ldots\ldots \quad (2)$$

The mean of (1) and (2) gives

$$F\beta m = \tfrac{1}{2}(U_{M1} + U_{M2})(\alpha\xi + \beta\eta + \gamma\zeta). \quad \ldots\ldots\ldots \quad (3)$$

The vertical component of the earth's field, V, is given by

$$V = F(\lambda l + \mu m + \nu n). \quad \ldots\ldots\ldots \quad (4)$$

Eliminating F from (3) and (4) gives

$$\tfrac{1}{2}(U_{M1} + U_{M2}) = \frac{V\beta m}{(\lambda l + \mu m + \nu n)(\alpha\xi + \beta\eta + \gamma\zeta)}.$$

Neglecting products of small quantities this gives

$$\tfrac{1}{2}(U_{M1} + U_{M2}) = V \cdot \frac{1}{\mu\eta}\left(1 - \frac{\lambda l}{\mu m} - \frac{\nu n}{\mu m} - \frac{\alpha\xi}{\beta\eta} - \frac{\gamma\zeta}{\beta\eta}\right). \quad \ldots\ldots \quad (5)$$

In practice the plane and the axis of vibration of the vibrating coil are adjusted so that α and γ are small by the process whereby U_{M1} is made nearly equal to U_{M2}. Also, the Helmholtz coil is adjusted so that ξ and ζ are very small.

We may, therefore, safely neglect the last two terms in equation (5). The simplified expression for V becomes

$$V = \mu\eta\left(\frac{U_{M1} + U_{M2}}{2}\right)\left(1 + \frac{\lambda l + \nu n}{\mu m}\right). \quad \ldots\ldots \quad (6)$$

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The angles $\cos^{-1}\lambda$ and $\cos^{-1}\eta$ can most certainly be adjusted to values smaller than 0·0001 radian. The values of μ and η will therefore differ from unity by a quite negligible amount.

The correction which has to be applied to the mean of the two U readings reduces to the simple term $(\lambda l + \nu n)/\mu m$ in relation to unity.

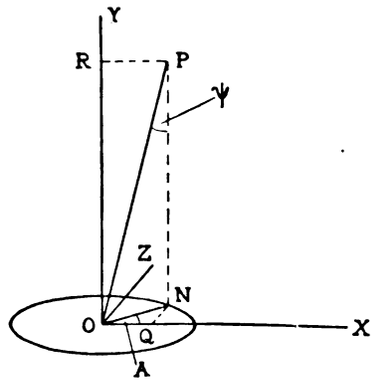

FIG. 5.

It is thus seen that the only quantity finally involved in these corrections arising from the adjustments of the magnetometer is the departure from verticality of the axis about which the Helmholtz coil and the vibrating coil are oriented.

The physical significance of the remaining correction terms is best seen by reference to Fig. 5.

OX, OY and OZ are the axes of reference, of which OY is the axis of turning. OP is the vertical of which the direction cosines λ, μ and ν are given by OQ, OR and NQ respectively.

The actual angle which the axis of turning makes with the vertical is ψ, and the vertical plane containing OY makes the angle A with the plane OXY.

We then have

$$\frac{\nu}{\mu} = \frac{\nu}{\sqrt{\nu^2 + \lambda^2}} \cdot \frac{\sqrt{\nu^2 + \lambda^2}}{\mu} = \tan\psi \sin A,$$

and $\lambda\mu = \tan\psi \cos A$.

Treating the earth's field F in the same manner as the vertical we have

$$\frac{n}{m} = \frac{\sqrt{n^2 + l^2}}{m} \cdot \frac{n}{\sqrt{n^2 + l^2}} = \frac{H}{V} \sin\beta, \qquad \frac{l}{m} = \frac{H}{V} \cdot \cos\beta.$$

On substitution of these terms into the expression

$$\nu n + \lambda l/\mu m,$$

the correction term becomes

$$\frac{H}{V} \tan\psi (\cos A \cos B + \sin A \sin B) = \frac{H}{V} \tan\psi \cos(A - B). \quad . \quad . \quad . \quad (7)$$

The maximum value which the correction can have occurs when the vertical plane containing the axis of rotation coincides with the vertical plane containing the earth's field, *i.e.*, A = B. The correction is then $H\psi/V$.

Expressed as an error in relation to unity, it is seen that the error may be large in regions near the magnetic equator, but expressed in terms of γ units the error is simply

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$H\psi$. For a maximum value of $H\psi$ equal to 0·5 C.G.S. unit, it is seen that ψ should be reduced to 2×10^{-5} radian for an accuracy of 1 γ to be obtained.

With the present instrument such an angle of departure of the axis of turning from the vertical can just be detected.

Appendix II.

Measurement of the Current.

The measuring circuit is as shown in Fig. 6.

R_1 is a standard four-terminal resistance shunted by the higher resistance R_2 of such value that a potential difference about 6 parts in 1000 greater than that of a standard cell results when the appropriate current flows through the combination.

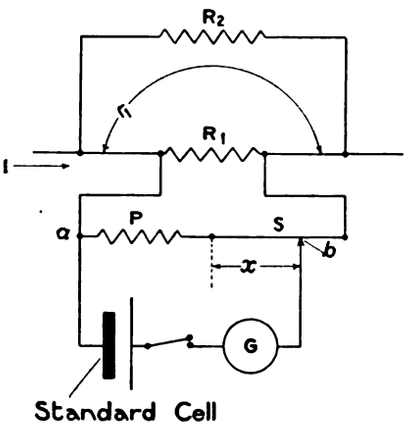

Fig. 6.

Since R_2 is shunted across the current terminals of R_1, it is necessary to know approximately the resistance of the current leads, shown as r_1.

Across the potential terminals of R_1 is shunted a relatively high resistance P in series with a slide wire of resistance S. The slide wire has a resistance about 0·005 that of P.

The standard cell of voltage E is connected through tapping key and galvanometer to the potential points a and b.

If a total current I is sent through this network, then the potential difference required is that between the points a and b, where b is at any reading x on the slide wire.

This potential difference is given by

$$\mathrm{PD} = \mathrm{I} \cdot \frac{\mathrm{P}+x}{\mathrm{P}+\mathrm{S}} \cdot \frac{\mathrm{R}_1\mathrm{R}_2}{\mathrm{R}_2 + \dfrac{\mathrm{R}_1(\mathrm{P}+\mathrm{S})}{\mathrm{R}_1+\mathrm{P}+\mathrm{S}} + r_1},$$

whence

$$\mathrm{I} = \mathrm{E} \cdot \frac{\mathrm{R}_1+\mathrm{P}+\mathrm{S}}{\mathrm{P}+x} \cdot \frac{\mathrm{R}_2 + \dfrac{\mathrm{R}_1(\mathrm{P}+\mathrm{S})}{\mathrm{R}_1+\mathrm{P}+\mathrm{S}} + r_1}{\mathrm{R}_1\mathrm{R}_2}.$$

The calibration of the scale of the slide wire S in terms of a small change in current is given by $\dfrac{d\mathrm{I}}{\mathrm{I}} = -\dfrac{dx}{\mathrm{P}+x}$.

In the case of the Helmholtz coil described, and for the value of V in England, the value of I is about 1·19 amperes.

It is convenient to use a slide wire of 1 ohm resistance and to choose a value of P

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such that the slide wire covers a range of about 5 parts in 1,000 in V. Under these conditions P becomes 200 ohms.

If $R_1 = 1$ ohm and $R_2 = 6$ ohms, then the effect of the current leads of R_1 is small, but not quite negligible, if they have a resistance of 0·0001 ohm.

For a base line corresponding to a value of x equal to 0·5 ohm the value of I becomes

$$1{\cdot}01830 + \frac{201}{200{\cdot}5} \times \frac{6 + \frac{201}{202}}{6} \text{ amp.}$$

$$= 1{\cdot}19017 \text{ amp.}$$

For a change of slide wire reading of 0·001 ohm the proportionate change of current necessary to give balance is equal to

$$\frac{0{\cdot}001}{200{\cdot}5} = 5 \text{ parts in } 10^6.$$

It is thus seen that accuracy to one part in a hundred thousand can be obtained and observed if R_1 and E are known to this accuracy.

A more open scale can be obtained on the slide wire at the expense of a smaller range by choosing a higher value of P. By introducing the constant of the Helmholtz coil into the calculations, it is easy to calibrate the scale of S to read change of V directly in γ units.

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XIV.

THE POWER LOSSES IN MAGNETIC SHEET MATERIAL AT HIGH FLUX DENSITIES

BY

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ABSTRACT.

From an examination of the results of a number of investigators it is shown that while there is general agreement as to the validity of STEINMETZ's expression ηB^x, where x is a constant index, for the hysteresis loss at flux densities up to about 10,000 C.G.S. units, there are considerable divergences between the results of different experimenters at flux densities above that value, particularly in tests on sheet material. Some find STEINMETZ's law to apply closely up to the highest flux densities at which tests can be carried out, whilst others record more rapid variations of hysteresis loss with B in the region of B = 10,000-16,000. The extent of the departure from STEINMETZ's law also varies considerably, some observers finding a maximum exponent of approximately 2·0 and others obtaining values as high as 3·2.

In the paper is described a series of tests, both by A.C. and ballistic methods, on a wide range of sheet materials. The results of these appear to confirm the increase in the exponent of B in the expression for the hysteresis loss at values of B between 10,000 and 15,000. They also show comparatively little difference between the various specimens of sheet material tested.

The construction of a new Lloyd square for tests at still higher flux densities, and a further more detailed series of tests, both A.C. and ballistic, on a representative specimen of each type of material, making use of this square, are then described. These tests were carried up to B = 17,000 or 18,000 and, while confirming and extending the previous results of increased exponents of B for values up to 16,000, they indicate a rapid decrease in the exponent at the highest values of B employed.

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THE POWER LOSSES IN MAGNETIC SHEET MATERIAL AT HIGH FLUX DENSITIES.

By C. E. WEBB, B.Sc., A.C.G.I., D.I.C., Assistant in the Electricity Department.

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I.—*Introduction.*

At the present time there is a marked tendency amongst designers of electrical machinery to employ higher flux densities in electrical sheet steel than have been customary in the past, and, in consequence, to demand guaranteed figures for the total power losses due to hysteresis and eddy currents at correspondingly higher densities. It is therefore a matter of increasing importance to obtain reliable and accurate knowledge of the way in which the power losses vary with the flux density at high densities, particularly in the special brands of sheet material which are now so extensively employed.

Of the two components—hysteresis and eddy currents—of which the total loss consists, the hysteresis loss is usually the more important, and is also the one concerning the law of variation of which there is more uncertainty. There is general agreement as to the validity of Steinmetz's empirical law that

$$\text{Hysteresis loss in ergs per cm.}^3 \text{ per cycle} = \eta B^x,$$

where x is a constant index for any given material, having a value of approximately 1·6, for flux densities between about 4,000 and 10,000 C.G.S. units; at higher densities, however, surprising differences between the results of different investigators appear to exist, as will be seen from the following brief survey of some of the principal papers bearing on the subject.

II.—*Summary and Discussion of Results of Other Investigators.*

Before proceeding to examine these results, reference should be made to a matter of terminology, in order to avoid an ambiguity which frequently arises in the discussion of laws of variation of the form to be considered when dealing with magnetic hysteresis. P. G. Agnew* pointed out that if, in an expression of the form $W = \eta B^x$, the index x is not a constant, but is itself a function of B, then the methods usually employed to

* P. G. Agnew, 'Bull. Bureau of Standards,' Vol. 7, Bull. 164, pp. 442-57, 1911.

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evaluate x, for any given range of B, do not give a correct result. These methods are the determination of the slope of the logarithmic curve connecting W and B, and the solution for x of the equations obtained by substituting two pairs of corresponding values of W and B, and AGNEW pointed out that when x is variable the quantity found in each case is not the true exponent, x, but the value of the function $[x + B \log B (dx/dB)]$. To this quantity he gave the name " ratio of variation," and he showed that it may be very different from the true exponent and may give very little indication as to the value of the latter. He also pointed out that the ratio of variation is in practice a much more convenient and useful quantity to employ than the true exponent, and that it has generally been used under the name of " exponent " by other investigators.

In view of this and of the fact that AGNEW's term is not generally familiar, it is proposed to adopt the term " exponent " throughout this paper when referring to the quantity described by AGNEW as the " ratio of variation." This use of the word is also justified by analogy with the case of a constant index, and at least for small ranges this quantity is the true exponent for a curve of constant index approximating to the actual variable-index curve.

Proceeding now to the consideration of published results bearing on the law of variation of hysteresis loss at high flux densities, the most direct treatment of the subject is that of F. STROUDE, in " An Accurate Examination of the Steinmetz Index for Transformer Iron, Stalloy and Cast Iron."* From tests on rings of each of these materials by the method of uniformly varying flux (slow cyclic) checked by ballistic tests he concludes that the Steinmetz index is practically constant for values of B above 4,000, the values found being 1·7 for transformer iron (tested up to B = 17,000), 1·66 for stalloy (up to B = 12,000), and 1·82 for cast iron (up to B = 6,000). It is to be noted, however, that the determination of the exponent is made from the slope of the logarithmic curve, and that owing to the compression of the curve above B = 10,000 when plotted in this way, the determination is not very accurate at these densities. In fact, on replotting the logarithmic curve on a large scale, there appears to be a distinct tendency for the exponent to increase for transformer iron to a value of about 1·9 or 2·0 between B = 10,000 and B = 17,000. This is confirmed if the exponent is found by solving for two points on the mean curve through the points obtained by plotting the hysteresis loss (W_H) and B. For stalloy the values found do not lie well on any smooth curve, and in any case do not extend much beyond the range for which STEINMETZ's law is known to apply fairly closely.

STROUDE refers to the work of a number of earlier investigators whose conclusions may be summarised as follows : STEINMETZ in his original paper,† working from EWING's results and from his own experiments, using A.C. wattmeter and magnetometer methods, found a practically constant index of approximately 1·6 from B = 2,000 to B = 20,000.

* F. STROUDE, ' Proc. Phys. Soc. Lond.,' Vol. 24, p. 238, 1912.

† STEINMETZ, ' Elec. Eng. N.Y.,' Vol. 10, p. 137, 1890 ; ' Trans. Amer. Inst. Elec. Eng.,' Vol. 9, p. 1, 1892.

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Ewing and Klaassen,* from ballistic tests, found that the exponent (deduced from the slope of the logarithmic curve) had a different value in each stage of the process of magnetisation, changing quite suddenly from one value to another. The value was approximately 2 for very low flux densities, fell to about 1·5 in the region of high permeability, and increased again to about 1·7 at high densities.

Both these experimenters used commercial soft iron such as was used in dynamo armatures—not at that time, of course, containing silicon or other special alloying constituents. Gumlich and Rose† tested rings containing various percentages of silicon by the ballistic method and also by alternating current, using a Franke curve-tracer. They assumed a constant index of 1·6 and calculated the corresponding value of the Steinmetz coefficient, η, for various densities. In every case η was found to increase from B = 10,000 to B = 16,000, in one case by as much as 25 per cent., corresponding to an increase in the index from 1·6 to 2·4. They also found that, assuming the eddy-current loss to vary as the square of the flux density, the coefficient of the eddy-current term decreased as the flux density increased, and that the eddy-current loss appeared actually to vary as the 1·8th or 1·9th power of the flux density. Prof. E. Wilson, V. H. Winson and G. F. O'Dell‡ carried out ballistic tests on lohys and stalloy sheets chiefly at low inductions, but recorded a few results between B = 6,000 and B = 13,500. These showed a marked tendency for the exponent to rise with increase of B, particularly in the case of stalloy. With both materials, however, the exponent exceeded 2 for the range of B = 11,500 to B = 13,500.

Lloyd and Fisher§ using the A.C. wattmeter method measured the total losses in both transformer steel and silicon steel specimens at two frequencies—30 and 60 cycles per second. They separated the total losses into hysteresis and eddy-current components by assuming these to be proportional to the first and second power of the frequency respectively, and found that for both types of material the exponent tended to increase at high flux densities, obtaining values ranging from 1·8 to 2·2 between B = 10,000 and B = 12,500. These measurements were made with great care and covered a large variety of specimens, and the results are probably the most reliable and accurate of those quoted by Stroude.

It will be seen that the investigators to whom Stroude refers obtained very divergent results, some in agreement with those recorded in his paper but others definitely in conflict with them. Before attempting to summarise the evidence for and against the constancy of the Steinmetz index, the author proposes to refer briefly to some other investigations of interest in this connexion.

Prof. F. G. Baily,‖ using a calorimetric method, deduced the hysteresis loss in a

* Ewing and Klaassen, 'Phil. Trans. Roy. Soc.,' A, Vol. 184, p. 985, 1893.

† Gumlich and Rose, 'Elektrotech. Zeits.,' Vol. 26, p. 503, 1905.

‡ Wilson, Winson and O'Dell, 'Proc. Roy. Soc.,' A, Vol. 80, p. 548, 1908.

§ Lloyd and Fisher, 'Bull. Bureau of Standards,' Vol. 5, Bull. 109, p. 453, 1909.

‖ F. G. Baily, 'Electrician,' Vol. 36, p. 116, 1895.

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laminated sample subjected to various alternating inductions up to B = 22,000. His figures indicate a gradual increase in the exponent to a maximum of about 2·4 in the region between B = 14,000 and B = 18,000, followed by a rapid and continuous decrease to a value slightly less than unity when B approaches 22,000. That this latter effect was not due to the H lines (which might be expected not to affect the hysteresis) becoming relatively a large proportion of the B, is shown by the fact that when the curve of hysteresis loss was re-plotted to a base of intensity of magnetisation $J = (B - H)/4\pi$, a similar tendency to bend over above about J = 1,400 was observed. This result, which does not seem to have been noticed by other investigators, is of considerable interest in view of the similar results recorded later in this paper.

Barrett, Brown and Hadfield,* in their comprehensive paper on the electrical conductivity and magnetic properties of a great variety of iron alloys, record ballistic tests on samples of annealed silicon iron (2·5 per cent. Si) and annealed Swedish charcoal iron, in which, assuming a constant index of 1·6, they found a considerable increase in the value of the coefficient η when B was raised to about 17,000. This increase amounted to nearly 20 per cent. for the silicon iron, corresponding to an exponent between B = 12,000 and B = 17,000 of approximately 2·1.

Czepek† in the course of a comparison between the losses under rotating and alternating magnetisation at various frequencies and inductions made a number of measurements of hysteresis loss by ballistic and A.C. wattmeter methods. Some of his ballistic results indicate a fairly constant index of about 1·9 for flux densities between 9,000 and 17,000, but he also records alternating-current results from which, assuming a constant hysteresis exponent of 1·6, he calculates the value of η and finds, in agreement with other investigators already quoted, that η increases considerably at high densities—in one case from 1·62 at B = 8,000 to 2·22 at B = 16,000. He also records in the same tests that, assuming the eddy-current losses to be proportional to the square of the frequency and the square of the density, the coefficient of the eddy-current term decreases very rapidly with increasing flux density—from 4·5 at B = 8,000 to 1·84 at B = 16,000.

Richter,‡ from ballistic and A.C. wattmeter results on a series of 7 rings of soft iron sheet, endeavoured to establish the law $W_H = aB + bB^2$, where a and b are constants, as giving a more accurate representation of the observed results than the Steinmetz formula. This, it will be seen, with increasing B, implies a continuous increase of the exponent, which, however, can never exceed 2. His results indicate a considerably smaller deviation from a law of this form than from a law involving a constant index, but it is noticeable on some of his curves that the experimental results at high densities, say B = 17,000, show an even more rapid increase of hysteresis loss than is given by Richter's law.

* Barrett, Brown and Hadfield, 'J. Inst. Elec. Eng.,' Vol. 31, p. 674, 1902.

† Czepek, 'Elektrotechnik u. Maschinenbau,' Vol. 28, pp. 325, 351, 1910.

‡ R. Richter, 'Elektrotech. Zeits.,' Vol. 31, p. 1241, 1910.

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L. W. WILD* made a large number of measurements of total losses on lohys and stalloy sheets of various thicknesses by the A.C. wattmeter method, using several different ways of building up the test-square. The tests were made at a number of values of B ranging from 2,500 to 15,000, and at 25 and 50 cycles per second. He did not separate the losses, but from his total-loss figures for any given B at two frequencies, the hysteresis loss may be readily calculated, assuming the proportionality of the hysteresis and eddy-current components to the first and second power of the frequency respectively, as was done by LLOYD and FISHER in their paper referred to above. The hysteresis values thus obtained show a slightly increasing exponent above B = 10,000, but the maximum value found does not exceed 2. Too much reliance must not be placed on the accuracy of these results, as the tests were made with the object of comparing the various methods of building up the square, and hence the attainment of identical conditions in each test, rather than high absolute accuracy, was aimed at.

A short note by W. J. WOOLDRIDGE† on " Hysteresis and Eddy-Current Exponents for Silicon Steel " bears directly on the question at issue and gives very interesting curves showing an increase of the hysteresis exponent from approximately 1·6 at flux densities below 7,000 to 3·2 at B = 16,000, and a decrease of the eddy-current exponent from 2·0 to 1·6 over the same range of B. Unfortunately, the value of the paper is much reduced, as no details of the method of testing (which was a wattmeter method) nor actual results from which the curves were deduced are given, but in the course of the discussion L. T. ROBINSON claimed to have obtained closely similar results by ballistic tests.

M. MACLAREN,‡ in an investigation primarily directed to finding the effect of temperature on the hysteresis loss in sheet steel, determined the hysteresis loss in samples of commercial armature steel and high silicon transformer steel at various flux densities between 6,000 and 14,000, from A.C. wattmeter measurements at 25 and 60 cycles per second, employing for the separation of the losses a method based on the usual assumption as to the variation of the two components with frequency, but involving the determination first of the coefficient of the eddy-current term. He determined the true exponent at each flux density, adopting as the coefficient of the hysteresis term the value found by assuming the exponent to be 1·6 at the lowest flux density employed. Under these conditions he found a nearly constant value for the true exponent, but, as was pointed out in the discussion, if the exponent is calculated by the usual methods from his results, values decreasing with increasing flux density are obtained in most cases, particularly at high temperatures. This result, which is in contradiction to most of those already recorded, is open to doubt owing to the absence of any allowance or correction for variations in the form factor, which would be almost certain to occur under the conditions of the test.

* L. W. WILD, ' J. Inst. Elec. Eng.,' Vol. 46, p. 217, 1911.

† W. J. WOOLDRIDGE, ' Trans. Amer. Inst. Elec. Eng.,' Vol. 30, p. 215, 1911.

‡ M. MACLAREN, *ibid.*, Vol. 30. p. 761, 1911.

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PAGLIANTI,* in a thorough investigation of the influence of silicon on the properties of steel, made a large number of measurements by the ballistic method of the hysteresis loss at flux densities from 7,000 to 16,000 on a series of annealed rods containing various percentages of silicon from 0·24 to 5·26. He determined for each value of hysteresis loss the value of the coefficient η, assuming an index of 1·6, and whilst he found great variations of the hysteresis, and hence of η, with changes in the percentage of silicon, he obtained in all cases very similar variations of η with B, viz., that η increased 25-50 per cent. as B was increased from 7,000 to 16,000, the increase becoming particularly rapid above B = 13,000. Values of the exponent between B = 13,000 and B = 16,000 ranged from 2·1 to 3·3.

A somewhat similar investigation of the properties of silicon-iron alloys containing various percentages of silicon was carried out by YENSEN† in the course of his comprehensive study of iron and iron alloys prepared by melting *in vacuo*. His measurements were made ballistically, using the Burrows permeameter, and he also found a very high exponent of B for the hysteresis loss at high values of B, particularly in the alloys containing large percentages of silicon. For one specimen the exponent between B = 10,000 and B = 15,000 was as high as 3·2. Considerable doubt has been thrown, however, on the accuracy of the Burrows method of testing for such highly permeable materials, and tests on rings of the same material were found to give much lower values of the exponent, the results not departing very widely from the Steinmetz relation.

One of the most interesting papers dealing with the losses in sheet material at high flux densities is that by J. S. NICHOLSON,‡ on " The Magnetisation of Iron at High Flux Densities with Alternating Currents." He points out that in testing total loss at high densities the chief difficulty is that, owing to the harmonics in the magnetising current, the wave-form of induced E.M.F., even with an alternator giving a good sine wave on open circuit, becomes very distorted, and that in consequence the true maximum B may be much less than that indicated by the r.m.s. induced E.M.F. To minimise the error from this cause he adopted a method which, although too complicated for general use, secures the elimination of the triple-harmonic component of the induced E.M.F. and of multiples thereof. To effect this he employs three similar cores, each with two magnetising windings, one set of windings being star-connected to a three-phase supply, and the other set mesh-connected to a single-phase alternator of triple frequency. Then by adjusting the phase and magnitude of the triple-frequency current until the voltage induced in three mesh-connected secondary windings and the power supplied by the single-phase machine are each reduced to zero, the third harmonic in the flux wave is eliminated and a sine wave of flux with only fifth, seventh and higher harmonics, which are usually very small, is obtained. Allowance was made for the fifth and seventh harmonics by analysing the E.M.F. waves, deducing the amplitudes of the flux harmonics

* P. PAGLIANTI, ' Métallurgie,' Vol. 9, p. 217, 1912 ; ' Rev. de Métallurgie,' Vol. 11, Extraits, p. 4, 1914.

† T. V. YENSEN, ' Trans. Amer. Inst. Elec. Eng.,' Vol. 34, p. 2601, 1915.

‡ J. S. NICHOLSON, ' J. Inst. Elec. Eng.,' Vol. 53, p. 248, 1915.

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from those of the induced E.M.F., and correcting the apparent value of the maximum density accordingly. A further correction was made for the flux in the air space within the secondary windings, using a permeability curve obtained ballistically.

The total losses in the three cores (built up of stalloy rings) were determined by measuring the three-phase input by a two-phase wattmeter at various values of the flux density (corrected as described above) and at frequencies of 15 and 22·5 cycles per second. The resulting values of total losses were separated into hysteresis and eddy-current losses by the usual method based on frequency variation, and the hysteresis losses obtained were plotted logarithmically against B. The curve approximated to a straight line, indicating a constant index of about 1·58, even up to B = 20,500. Prof. BAILY, however, analysing the results in a somewhat different way (by calculating the eddy-current loss at each flux density from the mean value of the eddy-current coefficient, instead of using the values found from the results at that flux density only) found that the logarithmic curve connecting hysteresis loss and B was markedly convex to the axis of B, *i.e.*, that the exponent increased with increasing B and rose to a value of approximately 3·5 above B = 20,000. The results recorded in the paper thus appear to be rather inconclusive, but they seem clearly to indicate that there is no tendency for the hysteresis exponent for stalloy to decrease even at inductions as high as 20,500—a result directly contradictory to that found by Prof. BAILY (see above), and, as he points out, hardly to be expected in view of the presumably constant value of the hysteresis loss when actual saturation is reached.

A recent investigation bearing on the variation of the iron losses at high flux densities is that carried out by JOUAUST and Mlle. BOURGOIGNON* at the Laboratoire Central de l'Électricité, on the losses in a large number of samples of sheet material at flux densities of 10,000 and 14,000. They used the Epstein tester as modified by ARMAGNAT, and separated the losses by the usual method based on frequency variation, using values of total losses at 25 and 50 cycles per second. They found that for some samples the exponent of B for hysteresis loss increased very much above the Steinmetz value—up to as high as 3·14 in one case—whilst for other samples considerably lower values, indicating an approximation to a square law of variation with B, were obtained. Their investigations appeared to show that the high values of the exponent were obtained on the group of specimens containing rather a higher percentage of silicon and having higher hysteresis loss and comparatively low permeability.

A still more recent paper bearing in some measure on the question under consideration is that by G. H. COLE† on "The Magnetic Properties of Silicon Steel in a Large Transformer." He measured the losses in the core of a 4,166-kva. single-phase transformer, built up of 4 per cent. silicon-steel sheets, by both a D.C. method using the volt-second meter as described by CHUBB and SPOONER,‡ and an A.C. method involving the

* JOUAUST and BOURGOIGNON, 'Bull. Soc. Franc. Elec.,' Vol. 3, p. 285, 1923.

† G. H. COLE, 'Electric J.,' Vol. 21, p. 17, 1924.

‡ CHUBB and SPOONER, 'Trans. Amer. Inst. Elec. Eng.,' Vol. 34, p. 2674, 1915.

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determination of the total losses at 25 and 60 cycles per second from wattmeter measurements and from oscillograms of magnetising current and voltage. The total losses found in the A.C. tests were separated into their two components by the usual frequency method, and the hysteresis losses thus obtained were found to be slightly greater than those given in the D.C. tests. The total losses measured on the transformer in this way were also found to be several per cent. higher than those found by Epstein tests on the material of the core, the discrepancy being attributed to inferior annealing of the core material, greater pressure in the core causing increased eddy currents, and, at high flux densities, fringing into the solid steel parts adjacent to the core. Although the hysteresis deduced from the A.C. measurements was slightly greater than that found from the D.C. tests, for both sets of tests the loss appeared to follow a similar law of variation with B. From the slope of the logarithmic curve connecting hysteresis loss with B, a constant exponent of approximately 1·6 was deduced for values of B up to about 9,000, but above this density the exponent was found to increase continuously and more and more rapidly with B up to a value of 5 at B = 16,000 (by A.C. tests), or at B = 18,000 (by D.C. tests). This abnormally high value was almost certainly due in part to leakage of the flux, owing to the reduced permeability at high densities, into adjacent solid steel parts of the transformer, but it was shown that the effect of this would be inconsiderable below B = 14,000, and a large increase in the exponent would therefore appear to be genuine. It is suggested that this may be due to the core material being unhomogeneous, *e.g.*, to the presence of oxide scale.

From the foregoing résumé it is clear that experimenters on the losses at high flux densities are very far from being unanimous as to the nature of the law connecting hysteresis loss with flux density. The results obtained fall roughly into the three classes set out below :—

(1) *Results confirming* STEINMETZ's *law*, in some cases even up to very high flux densities such as 20,000 or 22,000. In addition to the original results of STEINMETZ this group includes the work of STROUDE, MACLAREN, NICHOLSON and, in spite of divergences at lower densities, of EWING and KLAASSEN. In the case of STEINMETZ, EWING and KLAASSEN and, to some extent, STROUDE, as has already been pointed out, the tests were made on ordinary soft iron, not on the modern special brands of sheet material. It has also been suggested that STROUDE's results are not so conclusive as they were made to appear, whilst the tests on stalloy were only carried up to B = 12,000 ; nevertheless, with these reservations, they indicate at least a fairly close approximation to the STEINMETZ law. MACLAREN's results are, as previously stated, of doubtful accuracy, and in view of BAILY's analysis the reliability of NICHOLSON's figures for stalloy, at any rate at the highest densities, is open to question. The latter's tests, however, would certainly seem to indicate that at about B = 18,000 the hysteresis index does not differ very widely from the Steinmetz value.

(2) *Results indicating an exponent of approximately 2 at high inductions.* A large proportion of the investigators referred to above found that as the flux density was

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increased the hysteresis loss tended to rise rather more rapidly than was to be expected from STEINMETZ's law—some obtaining an exponent of almost exactly 2, others slightly higher values such as 2·1, 2·2 or even 2·4. Of this group, the results of WILD, as has been indicated above, cannot be considered very reliable, whilst those of WILSON, WINSON and O'DELL and of BAILY were limited to one or two specimens. The remaining results in this class, however—those of BARRETT, BROWN and HADFIELD, GUMLICH and ROSE, CZEPEK, RICHTER, and LLOYD and FISHER—were obtained on a variety of specimens and do not seem open to any serious criticism as regards the accuracy of the methods employed.

(3) *Results indicating still higher exponents, ranging up to* 3 *or more.* Several experimenters have found, over certain ranges of B, or for certain selected specimens, values of the exponent of 3 or more. These are WOOLDRIDGE, PAGLIANTI, YENSEN, JOUAUST and COLE. Of these, the results published by WOOLDRIDGE are not given in sufficient detail to carry much weight, those of YENSEN were obtained by a method of doubtful reliability when used for specimens with the properties of those tested by him, whilst those of COLE, being obtained on a complete transformer instead of on standard samples under exactly specifiable conditions of test, are subject to various sources of error and cannot be considered of very high accuracy. The tests of PAGLIANTI and JOUAUST, however, were carried out by standard methods and on a large number of specimens, and, although the values found for the exponent differed considerably for different specimens, all gave very high values at high densities. Both these investigators obtained values of the exponent for silicon iron ranging up to about 3·2—in PAGLIANTI's tests for the range from B = 13,000 to B = 16,000, in JOUAUST's experiments for the range from B = 10,000 to B = 14,000.

Whilst, therefore, there is a large body of evidence that, at least for silicon-iron sheets, the losses increase rather more rapidly with the flux density at densities above 10,000 than at densities below that value where STEINMETZ's law holds true, there is considerable disagreement between different investigators as to the extent of the deviation from the 1·6th power law, whilst two or three experimenters obtain results indicating that no such deviation occurs.

III.—*Preliminary General Investigation.*

Object and Scope of Tests.—The object of the tests to be described in this paper was to investigate the reasons for these apparently discordant results and to determine the true nature of the variation of the power losses with flux density at high densities.

In order to discover to what extent the law of variation differed for different commercial sheet materials or for different specimens of the same material, tests were made on a series of specimens provided by Messrs. J. Lysaght, Ltd., comprising samples of various thicknesses of their lohys, special lohys, high-resistance and medium-resistance grades. In each case tests were carried out at a number of values of B between 5,000 and 15,000. To provide a check against systematic errors inherent in the method of testing, parallel

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tests were made throughout by the ballistic and the A.C. wattmeter methods, the total power losses measured by the latter method being separated into hysteresis and eddy-current components, and the hysteresis thus deduced being compared with the static value obtained ballistically.

Methods of Testing.—(*a*) *Alternating-current Tests.*—The alternating-current tests followed the usual lines of the wattmeter method of measuring total iron losses. A diagrammatic representation of the circuit for this is shown in Fig. 1, and the general

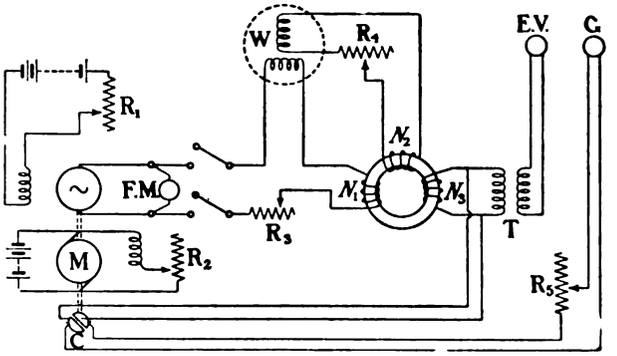

FIG. 1. A.C. Testing Circuit.

principles of the method have been frequently described, for example, by A. CAMPBELL* in his paper: " On the Magnetic Testing of Iron with Alternating Current."

The samples used in these tests consisted of flat strips. Half of these were cut parallel and half perpendicular to the direction of rolling. They were built up into a square of the pattern due to LLOYD and FISHER,† with certain modifications. One of the most important of these was an increase in the depth of windings to allow strips 7 cms. wide to be inserted in place of the 5-cm. strips used in the original apparatus. This reduces the effect on the results obtained of the higher losses in the material near the edges, caused by the cutting of the samples.

The square was wound with two primary (magnetising) coils of 640 and 160 turns respectively, and with secondary coils of 160, 80, 40 and 20 turns, which could be connected to the wattmeter pressure circuit or to the voltmeters as required, to give convenient deflections. The square is indicated purely diagrammatically in Fig. 1, N_1, N_2, N_3 representing the three windings actually in use.

The wattmeter, W, was of the reflecting dynamometer type, designed to be of high sensitivity, and constructed largely of marble to avoid eddy currents and thereby give accurate results at high frequencies. It had been found reliable up to 800 cycles per second. A variable resistance, R_4, in series with the pressure coil gave a wide range of multipliers.

* A. CAMPBELL, 'N.P.L. Collected Researches,' Vol. 6, Paper 7, p. 77 (or 'J. Inst. Elec. Eng.,' Vol. 43, p. 553, 1909).

† LLOYD and FISHER, *loc. cit.*

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To determine the value of $B_{max.}$ it is necessary to measure the mean voltage induced in the secondary winding N_3, since

$$V_{mean} = 4fN_3B_{max.}A \times 10^{-8},$$

where f = frequency,

N_3 = number of turns in secondary winding,

A = cross-section of specimen.

For this purpose a Cambridge and Paul bifilar moving-coil reflecting galvanometer G with a high adjustable series resistance R_5 was used, connected in series with a synchronous commutator, C.

An electrostatic voltmeter, EV, connected to the same or another secondary winding, either directly or through a variable-ratio step-up transformer T, measured the r.m.s. voltage and hence, in conjunction with the galvanometer G, gave the form factor of the induced E.M.F. wave. The voltmeter used was a low-reading Ayrton-Mather electrostatic instrument having a working range of 3–8·5 volts.

The source of supply was an alternator of about 5 kW rating, which gave a close approximation to a sine wave of voltage on open circuit, and the frequency and excitation of which were readily adjustable by means of the resistances R_1 and R_2. Its frequency was measured by a reed frequency-meter, FM, connected permanently to the supply terminals.

Before making any measurements the wattmeter and moving-coil galvanometer were always calibrated, as they were found to be liable to slight variations in sensitivity. To calibrate the wattmeter a direct current of exactly 1 ampere was passed through the current coil and through a standard resistance of 1 ohm, the potential terminals of which were connected to the pressure coil. A power of exactly 1 watt was thus applied to the instrument and the deflection noted. The current was then reversed and the deflection again noted, the mean of the two deflections being taken as the A.C. calibration. The galvanometer was similarly calibrated by connecting it to the potential terminals of the 1-ohm resistance. To adjust the current accurately to 1 ampere it was passed through a standard resistance of 1·0183 ohm, the P.D. on which was balanced against the E.M.F. of a Weston standard cell. This adjustment could easily be made to 1 part in 10,000.

The electrostatic voltmeter was not found to vary appreciably in its sensitivity and was therefore not regularly calibrated. It was checked at intervals on a potentiometer against a Weston standard cell.

In taking a reading, after setting the frequency to the required value, the alternator excitation was first adjusted to give approximately the desired flux density, and the brushes on the commutator were slowly rotated to give the maximum reading on the mean-voltage galvanometer. The final adjustment of this reading to the value corresponding to the desired flux density was made by means of the alternator field resistance, and the wattmeter and electrostatic voltmeter were then read.

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In order to separate the losses into hysteresis and eddy-current components, readings were taken at the same values of B at two frequencies—25 and 50 cycles per second. Then, if W_1 and W_2 are the total losses at frequencies f_1, f_2, for a given value of B, we may write :—

$$W_1 = \eta f_1 B^x + \xi f_1^2 k^2 B^y$$

$$W_2 = \eta f_2 B^x + \xi f_2^2 k^2 B^y$$

$$= 2\eta f_1 B^x + 4\xi f_1^2 k^2 B^y \quad (\text{if } f_2 = 2f_1),$$

whence

$$4W_1 - W_2 = 2\eta f_1 B^x = 2H_1 = H_2,$$

if H_1, H_2 = hysteresis losses at frequencies f_1, f_2.

It is clear that this relation only holds if the form factor, k, is the same at both frequencies. This is not usually the case, particularly at high flux densities, unless special steps are taken to secure this condition. Consequently, in these tests all values of total loss were reduced to a sine wave of secondary induced voltage. For this purpose, for any values of B and frequency for which the form factor departed appreciably from the sine-wave value, a series of readings was taken with the form factor artificially varied by the insertion of series resistance R_3 in the magnetising circuit. Then from the above equations it is clear that for given values of B and f the curve connecting W and k^2 is a straight line, and by extrapolating to the value $k^2 = (1{\cdot}11)^2 = 1{\cdot}232$ the total losses for a sine wave of induced E.M.F. can be obtained. Using the values thus found, the above formula may be legitimately applied to separate the losses.

A correction on the apparent flux density must also be made for the flux in the air space enclosed by the secondary winding employed to measure B. Owing to the existence of this flux the value of B calculated from the mean voltage induced in the winding is too large by an amount equal to

$$H \times \frac{\text{Area of air space}}{\text{Section of specimen}} = H \times \frac{\text{Mean area enclosed by coil} - \text{Section of specimen}}{\text{Section of specimen}}.$$

The multiplying factor, being a constant for any given specimen and square, is calculated once for all from the known dimensions of the winding and specimen. The value of H for a given B may be taken from the permeability curve obtained ballistically, or may be directly measured by alternating current, as the results obtained by the two methods do not differ appreciably (see Appendix I, p. 310). This correction becomes important only above B = 10,000.

The values of loss obtained in the tests must also be corrected for the power lost in the instruments. The deduction to be made for the loss in the wattmeter pressure circuit is $V^2_W R_{P.C.}/$, where

$$V_W = \frac{N_2}{N_3} \times \text{r.m.s. voltage indicated on winding } N_3,$$

and

$$R_{P.C} = R_4 + \text{resistance of pressure coil itself.}$$

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This correction was always very small, particularly at high flux densities.

The loss in the moving-coil galvanometer circuit was in all cases negligible owing to the very high value of the series resistance R_5. It was usually less than 0·01 per cent. of the power being measured.

Allowance for the reduction in the flux density caused by the overlapping of the specimen and the corner pieces was made in the way suggested by LLOYD and FISHER,* and in accordance with their formula, *i.e.*, the loss per kg. of material was determined by dividing the measured loss by an effective mass m given by

$$m = (m_1 + m_2)(1 - 1{\cdot}4c),$$

where

m_1, m_2 = masses of specimen and corner pieces,

c = total length of overlapping portions of magnetic circuit, expressed as a fraction of the total length of the circuit.

When corner pieces of different material from the sample were used, an additional correction for the difference between the losses in the corner and the sample was necessary. This was made in accordance with the formula

$$W = W_1 + \frac{m_2}{m_1}(W_1 - w),$$

where

W = true loss in watts per kg.

W_1 = apparent loss in watts per kg.,

w = loss in corner pieces in watts per kg.

This formula, though only approximate, was sufficiently accurate provided the corner pieces resembled the test-specimen fairly closely in density, thickness and specific power loss. Where corners were cut from the same sheet as the specimen no such correction was made, it being known that the effect of the hardening due to bending of the corners was not serious.

(*b*) *Ballistic Tests.*—The ballistic determinations of the hysteresis loops were made by the standard method, which is too well known to need any description. The test-set employed was the same, except for very minor alterations in certain details, as that described at length in Dr. DYE's article on "Magnetic Measurements and Properties of Materials," in GLAZEBROOK's 'Dictionary of Applied Physics.'†

The ballistic galvanometer was a Sullivan moving-coil instrument having a resistance of approximately 600 ohms and a natural period in the field of about 10 seconds. It was

* LLOYD and FISHER, *loc. cit.*, p. 463.

† GLAZEBROOK, 'Dictionary of Applied Physics,' Vol. 2, pp. 471–474. Macmillan, London, 1922.

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fitted with adjustable shunts and series resistance to allow of the instrument being made direct-reading for B on any specimen. It was calibrated by being connected to the secondary of a Campbell inductometer in the primary of which a current of exactly 1 ampere was suddenly reversed. The current of 1 ampere was measured by a potentiometer and Weston standard cell as described above under "Alternating-current Tests."

The magnetising current was measured by a Paul bifilar moving-coil galvanometer fitted with a range of suitable shunts, and adjusted by varying a large series resistance to give direct readings of H. This instrument was calibrated by means of a Weston standard millivoltmeter with standard shunts.

The ballistic tests were made on rings of external diameter 15 cms. and internal diameter 12 cms. cut from the same sheets as the strips used for the A.C. measurements. It was realised that owing to the small width of these rings—1·5 cms. as compared with 7 cms. for the strips—the effect of cutting would be much greater than on the strips, and that the values of hysteresis would, therefore, probably not be the same in both sets of tests; but as the investigation was directed rather to the nature of the variation of loss with B than to the determination of the actual values of the losses, it was thought that tests on the rings would furnish the confirmation of the A.C. results which was required.

The static hysteresis loops were plotted from the corresponding values of H and B, and the hysteresis loss was calculated from their areas (measured by a planimeter) in the ordinary way and expressed in watts per kg. at 50 cycles per second.

Results Obtained.—A complete list of the materials tested is given in Table I. From the sheets of each material, sets of strips and rings, for A.C. and ballistic tests respectively, were cut as described above. The former were in each case denoted by the number of the material and the letter A, the latter by the number and the letter B.

TABLE I.—Materials Tested.

No.	Material.	Density.	Measured thickness.
			mm.
1	Lohys	7·85	0·96
2	Lohys	7·85	0·46
3	Lohys	7·85	0·43
4	Lohys	7·85	0·37
5	Special lohys	7·80	0·45
6	Special lohys	7·80	0·40
7	High resistance	7·56	0·46
8	High resistance	7·56	0·35
9	High resistance	7·56	0·34
10	High resistance	7·56	0·32
11	Medium resistance ...	7·76	0·47
12	Medium resistance ...	7·76	0·38
13	Medium resistance ...	7·76	0·37
14	Medium resistance ...	7·76	0·37

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As an example of the A.C. results obtained, the complete results for specimen 4A are given in Table II. It will be seen that up to B = 12,000 the form factor, k, of the wave of induced E.M.F. was practically that for a sine wave, but above this flux density it was not possible to obtain a sinusoidal wave and the losses for such a wave-shape were found by extrapolation on the curve connecting the wattmeter reading and k^2.

TABLE II.—A.C. Results on Specimen 4A.

Mass $m_1 = 1.232$ gms. Cross-section of specimen, $A = 1{\cdot}569$ cm.^{2}. Mass of corners, $m_2 = 66$ gms. Overlap at corners $= 4{\cdot}5$ mms. Effective mass, $m = 1{,}229$ gms. Magnetising turns, $N_1 = 640$. Secondary turns (wattmeter), $N_2 = 160$.

Section of secondary coil = 20 cm.^{2}, therefore correction for air flux
$= [(20 - 1{\cdot}569)/1{\cdot}569]\ H = 11{\cdot}8\ H.$

$B_{max.}$ (apparent).	Voltmeter turns, N_3.	$V_{mean.}$	$V_{r.m.s.}$	k.	k^2.	Total watts (corrected).	Watts / kg.	$B_{max.}$ (corrected).
			Frequency f = 25 cycles per sec.					
5,000	80	0·628	0·696	1·108	1·228	0·468	$0{\cdot}38_1$	4,980
10,000	40	0·628	0·698	1·111	1·234	1·514	$1{\cdot}2_3$	9,960
12,000	40	0·753	0·841	1·116	1·245	2·175	$1{\cdot}7_7$	11,930
14,000	20	0·439	0·503	1·145	1·312	3·090		
			0·525	1·198	1·435	3·141		
			0·572	1·303	1·696	3·253		
			Sine wave	1·110	1·232	3·058	$2{\cdot}4_9$	13,880
16,000	20	0·502	0·619	1·233	1·520	4·328		
			0·693	1·381	1·906	4·524		
			0·762	1·517	2·301	4·727		
			Sine wave	1·110	1·232	4·181	$3{\cdot}4_0$	15,730
			Frequency f = 50 cycles per sec.					
5,000	40	0·628	0·696	1·108	1·228	1·114	$0{\cdot}9_1$	4,980
10,000	20	0·628	0·698	1·111	1·234	3·651	$2{\cdot}9_7$	9,960
12,000	20	0·753	0·839	1·115	1·243	5·181	$4{\cdot}2_3$	11,930
14,000	20	0·879	1·003	1·140	1·299	7·291		
			1·038	1·180	1·394	7·419		
			1·085	1·235	1·524	7·658		
			Sine wave	1·110	1·232	7·179	$5{\cdot}8_4$	13,880
16,000	20	1·004	1·213	1·208	1·459	10·23		
			1·375	1·369	1·874	11·00		
			1·548	1·542	2·378	12·00		
			Sine wave	1·110	1·232	9·80	$7{\cdot}9_7$	15,730

The method of deducing the corrections to be applied to the apparent values of B is shown in Table IIA, the values of H being derived from the permeability curve shown in Fig. 2.

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TABLE IIA.—Correction for Air Flux for Specimen 4A.
Correction to B for air flux = 11·8H (see Table II).

B (apparent).	H (actual).	H correction on B.	B (corrected).
5,000	1·4	20	4,980
10,000	3·5	40	9,960
12,000	6	70	11,930
14,000	10	120	13,880
16,000	23	270	15,730

The total losses obtained have been plotted against the corrected flux densities in Fig. 3, and from these curves the values used in the separation of the losses have been taken. The separation is carried out in Table III, which also contains the hysteresis

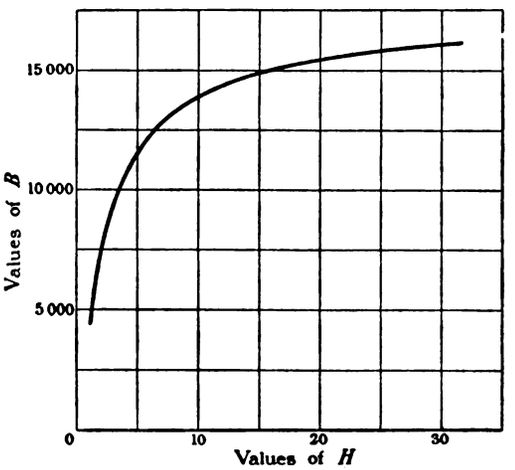

FIG. 2. Permeability Curve on Specimen No. 4.

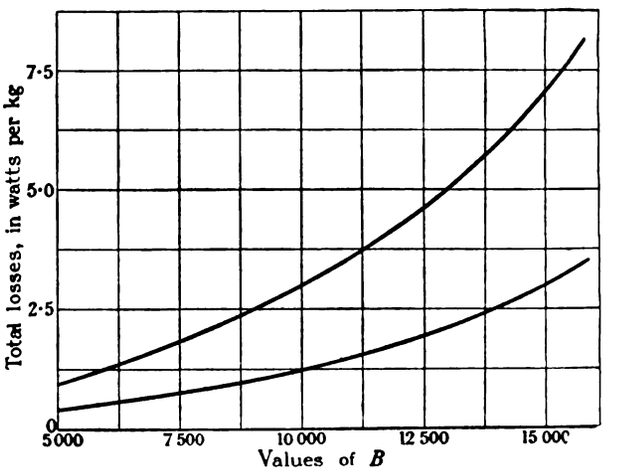

FIG. 3. Curves Connecting Total Losses and B for Specimen No. 4.

TABLE III.—Separation of Losses for Specimen 4A.

B.	W_1 (at 25 cycles per sec.).	W_2 (at 50 cycles per sec.).	$H_2 = 4W_1 - W_2$.	$E_2 = W_2 - H_2$.	H_2 (from ballistic tests).	Exponents.		
						Hysteresis.		Eddy-current.
						A.C.	Ballistic.	
5,000	0·38	0·91	0·61	0·30	0·62			
						1·68	1·73	1·8
10,000	1·24	3·00	1·96	1·04	2·06			
						2·11	2·05	1·58
12,500	1·94	4·62	3·14	1·48	3·26			
						2·53	2·19	1·87
15,000	3·01	7·06	4·98	2·08	4·86			

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losses found by ballistic tests on specimen 4B and the exponents calculated between each pair of successive values of B both for hysteresis and eddy-current losses.

In Table IV the results are given of a similar analysis of the measurements made, by alternating current and ballistically, on the remaining specimens detailed in Table I. In certain cases, however, only two or three ballistic tests were carried out, and determinations of the exponent for the same ranges of B as in the A.C. tests could not be made.

TABLE IV.

No.	$B_{max.}$	Hysteresis loss in watts/kg. at 50 cycles per sec.		Exponent of B for hysteresis loss.		No.	$B_{max.}$	Hysteresis loss in watts/kg. at 50 cycles per sec.		Exponent of B for hysteresis loss	
		From A.C. tests.	From ballistic tests.	From A.C. tests.	From ballistic tests.			From A.C. tests.	From ballistic tests.	From A.C. tests.	From ballistic tests.
1*	5,000	0·70	0·60			5	5,000	0·62	0·64		
				1·72	1·79					1·57	1·67
	10,000	2·30	2·07				10,000	1·84	2·04		
				1·25						1·75	2·00
	12,500	3·04					12,500	2·72	3·18		
										2·86	2·18
	15,000						15,000	4·58	4·73		
2	5,000	0·61	0·59			6	5,000	0·62			
				1·62	1·65						
	10,000	1·87	1·85				10,000	1·91	2·12	1·61	
				2·09	1·95						
	12,500	2·98	2·86				12,500	3·05		2·10	
				2·29	2·06						
	15,000	4·52	4·21				15,000	5·00		2·84	
3	5,000	0·64				7	5,000	0·42	0·42		
				1·61						1·66	1·67
	10,000	1·96	1·87				10,000	1·33	1·34		
				1·94						1·79	1·86
	12,500	3·02					12,500	1·98	2·03		
				2·62						2·41	2·26
	15,000	4·87					15,000	3·07	3·06		
4	5,000	0·61	0·62			8	5,000	0·39			
				1·68	1·73					1·74	
	10,000	1·96	2·06				10,000	1·30	1·54		
				2·11	2·06					2·24	
	12,500	3·14	3·26				12,500	2·14			
				2·53	2·19					2·51	
	15,000	4·98	4·86				15,000	3·38			

* Owing to the thickness of the laminations of this sample, the eddy-current loss was very large and the temperature rise so rapid at high inductions that reliable results could not be obtained. This also accounts for the apparently low value of hysteresis at B = 12,500.

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TABLE IV (*continued*).

No.	$B_{max.}$	Hysteresis loss in watts/kg. at 50 cycles per sec.		Exponent of B for hysteresis loss.		No.	$B_{max.}$	Hysteresis loss in watts/kg. at 50 cycles per sec.		Exponent of B for hysteresis loss.	
		From A.C. tests.	From ballistic tests.	From A.C. tests.	From ballistic tests.			From A.C. tests.	From ballistic tests.	From A.C. tests.	From ballistic tests.
9	5,000	0·41				12	5,000	0·62	0·69		
	10,000	1·28	1·38	1·64			10,000	1·90	2·11	1·61	1·61
	12,500	2·01		2·03			12,500	2·86	3·14	1·83	1·78
	15,000	3·10		2·38			15,000	4·63	4·78	2·64	2·31
10	5,000	0·36	0·43			13	5,000	0·62			
	10,000	1·18	1·37	1·70	1·65		10,000	1·90	2·18	1·62	
	12,500	1·81	2·05	1·93	1·82		12,500	2·95		1·97	
	15,000	2·97	3·09	2·70	2·26		15,000	4·71		2·57	
11	5,000	0·67	0·68			14	5,000	0·60	0·67		
	10,000	2·00	2·15	1·58	1·66		10,000	1·79	2·05	1·58	1·61
	12,500	3·08	3·27	1·90	1·88		12,500	2·78		1·97	
	15,000	4·84	4·93	2·48	2·26		15,000	4·72		2·91	

Discussion of Results of Preliminary General Investigation.—These results appear to show definitely that above B = 10,000 the exponent of B for hysteresis loss rises much above the Steinmetz value, 1·6, which applies below B = 10,000. In several cases the exponent for the range B = 12,500 to 15,000 approximates to or even exceeds 2·5, and it would seem likely that, for smaller ranges of B near 15,000, values as high as 3·0 might be obtained. Such values were actually obtained for the range B = 12,500 to 15,000 for material which was known to be defective. It will be seen that in all cases a fairly close agreement is obtained between the A.C. and ballistic tests. The increase in the exponent is quite definite in the ballistic results, though less marked owing to the hysteresis loss at flux densities near 10,000 (the region of maximum permeability) being increased much more than at higher flux densities by the greater effect of cutting on the narrow rings used in the ballistic tests. That this is the correct explanation of

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the lower exponents obtained in the ballistic tests was proved by a complete series of tests carried out on annealed rings and strips (see Appendix II) for which close agreement between the two methods of test was obtained.

The results for the four kinds of sheet material tested do not show any pronounced differences in the manner of variation of the hysteresis loss with B, the exponent increasing to approximately the same extent with B for each material. The increase appears to be slightly less than the average for lohys, and slightly greater for special lohys specimens, but the number of specimens tested was too few to establish the existence of so small a difference.

The methods adopted in this preliminary general investigation were directed rather to obtaining comparative results on a number of specimens readily than to securing high absolute accuracy in any individual test. Thus the method of separating the total losses into hysteresis and eddy-current components assumes independence of frequency for the hysteresis loss per cycle, which, although a fairly close approximation, probably introduces some slight error into the A.C. values of hysteresis. At high flux densities, in particular, the results are liable to considerably increased errors, since the corrections to be applied for air flux and for form factor of E.M.F. wave become very large.

In view of the very large increase already observed in the exponent at high flux densities, it becomes of interest to examine the effect at still higher flux densities, since, as no further increase of hysteresis is to be expected when saturation is reached, it would seem that the ratio of variation must at some point begin to diminish with increased B.

It was therefore decided to make a more accurate and detailed examination of the variation of losses on certain representative specimens, and to carry this examination to higher values of B.

IV.—*More Detailed Examination of Certain Specimens.*

Construction of New Square.—To allow of tests being made at higher flux densities, a new Lloyd square was constructed in which, using the same general disposition and numbers of turns in the windings, the following improvements were introduced :—

(1) Cross-section of secondary coil was reduced from approximately 20 cm^2. on the old square to 8·5 cm.^{2}, thereby reducing the air-flux correction to 40 per cent. of its previous value. This is very important for tests at high flux densities.

(2) Resistance of primary winding was greatly reduced by winding with very much thicker wire. This reduces the distortion of the E.M.F. wave and hence the correction to be applied for departure of form factor from a sine-wave value. It is also of importance in diminishing the I^2R heating due to the magnetising current, which is liable to become considerable at high flux densities and to affect the temperature of the specimen.

Method of Testing and Working Out Results.—Using this square, a further complete series of tests was made on specimens Nos. 3, 6, 8, 12, *i.e.*, one each of lohys, special lohys, high-resistance and medium-resistance material.

As in the first part of the investigation, parallel tests were made throughout by the

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A.C. and the ballistic methods. The A.C. tests were carried out as before at frequencies of 25 and 50 cycles per second and at a number of flux densities from 10,000 to the highest value that could be reached without excessive heating. Owing to the improved characteristics of the new square, it was possible to obtain satisfactory results for flux densities up to about 18,000 on the softer materials and 17,000 on the high-resistance material.

At the highest values of B attained, the I^2R heating, even with the new square, was sufficient to produce an appreciable rise of temperature in the time required to make the adjustments and take the readings. To minimise the effect of this, the temperature of the samples was read by a thermo-junction and the square was allowed to cool between successive readings so that the temperature when the readings were taken did not vary more than 1° or 2°.

The separation of the losses was carried out independently at each frequency from the curve connecting the power loss and the square of the form factor. This avoids the assumption that the hysteresis loss per period is independent of the frequency, but assumes that it is independent of form factor. From the fundamental formula

$$W = \eta f B^x + \xi f^2 k^2 B^y$$

it follows that if f and B are kept constant,

$$W = A + Bk^2,$$

i.e., the curve connecting W and k^2 should be a straight line.

In carrying out the tests a series of readings at different form factors was taken at each value of B by inserting various series resistances in the magnetising circuit, and curves connecting W/f and fk^2 were drawn. (These curves are identical with curves connecting W and k^2, except for a change of scale, for any series of tests at a given B and frequency, but they allow a direct comparison to be made between the eddy-current coefficients found from the 25- and 50-cycle tests.) These curves were found to approximate very accurately to straight lines, suggesting that the assumption of independence of form factor for the hysteresis loss is at least a close approximation to the facts. The slope of each of these curves was then found and must evidently be equal to ξB^y. Plotting the slopes logarithmically against the corresponding values of B, a curve is obtained the slope of which is the exponent of B for the eddy-current loss. In order to eliminate the errors of individual tests, the separation of the losses was carried out by deducing from the curve the eddy-current loss for a sine-wave form for each value of B, and then calculating the hysteresis loss by subtracting this value from the total loss for a sine-wave form, determined, as previously described, by extrapolation on the curve connecting W/f and fk^2.

The ballistic tests were carried out exactly as before, except that the tests were made on the same strips as were used for the A.C. tests while still built up in the Lloyd square instead of on separate ring specimens. The square was then treated as a ring, the magnetising force being calculated from the primary current, and the flux density being measured by using one of the secondary windings as a search coil. This eliminates any

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discrepancy due to different effects of cutting, but may lead to some inaccuracy in the measurement of H owing to the magnetic circuit not being perfectly uniform. Comparative tests, however, showed that the error due to this was quite small (see Appendix III).

Comparable values of hysteresis loss measured ballistically and by alternating current at 25 and 50 cycles per second were thus obtained entirely independently and afforded a double check on the variation of loss with B.

Results obtained.—As in the first part of the investigation the complete A.C. results for one sample, No. 8A, are given in Table V, the calculation of the air-flux correction to the apparent value of $B_{max.}$ being shown in Table VA.

TABLE V.—A.C. Results on Specimen 8A (High-Resistance Material).

Mass $m_1 = 1{,}108$ gms. Section of specimen, $A = 1{\cdot}465$ cm.^{2}. Mass of corners, $m_2 = 62$ gms. Overlap at corners $= 4{\cdot}3$ mms. Effective mass, $m = 1{,}113$ gms.
Magnetising turns, $N_1 = 640$. Secondary turns (wattmeter), $N_2 = 160$.
Cross-section of secondary coil $= 8{\cdot}5$ cm.^{2}, therefore correction for air flux $= [(8{\cdot}5 - 1{\cdot}465)/1{\cdot}465]H = 4{\cdot}8H$.

$B_{max.}$ (apparent).	$V_{mean.}$	$V_{r.m.s.}$	k.	fk^2.	Total watts (corrected).	Watts/kg. = W.	$\frac{W}{f}$.	$B_{max.}$ (corrected).
(1) Frequency $f = 25$ cycles per sec. Secondary turns (voltmeter), $N_3 = 40$.								
10,000	0·586	0·658	1·124	31·6	0·765	0·687	0·02748	9,990
		0·726	1·239	38·4	0·780	0·701	0·02804	
		0·806	1·375	47·3	0·798	0·717	0·02868	
		0·844	1·440	51·8	0·804	0·722	0·02888	
						Sine wave	0·02743	
12,000	0·703	0·795	1·131	32·0	1·113	1·000	0·040	11,970
		0·846	1·204	36·3	1·129	1·014	0·04064	
		0·950	1·351	45·6	1·154	1·036	0·04144	
		1·065	1·515	57·4	1·187	1·067	0·04268	
						Sine wave	0·0398	
14,000	0·820	0·96	1·17	34·2	1·592	1·430	0·0572	13,900
		1·07	1·305	42·6	1·625	1·460	0·0584	
		1·198	1·46	53·3	1·667	1·498	0·05992	
		1·355	1·652	68·3	1·723	1·547	0·06188	
						Sine wave	0·0567	
15,000	0·879	1·093	1·243	38·6	1·936	1·739	0·06956	14,790
		1·218	1·385	48·0	1·979	1·776	0·07104	
		1·245	1·416	50·2	1·986	1·784	0·07136	
		1·435	1·632	66·6	2·054	1·845	0·07380	
						Sine wave	0·0684	
16,000	0·938	1·23	1·311	42·9	2·213	1·990	0·0796	15,650
		1·368	1·459	53·2	2·261	2·031	0·08124	
		1·478	1·575	62·0	2·308	2·072	0·08288	
		1·53	1·63	66·5	2·331	2·095	0·0838	
						Sine wave	0·0774	
17,000	0·996	1·378	1·383	47·8	2·431	2·184	0·08736	16,480
		1·463	1·469	54·0	2·467	2·214	0·08856	
		1·50	1·506	56·7	2·490	2·237	0·08948	
		1·575	1·582	62·6	2·522	2·267	0·09068	
						Sine wave	0·0838	

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Table V. (*continued*).

$B_{max.}$ (apparent).	$V_{mean.}$	$V_{r.m.s.}$	k.		Total watts (corrected).	$\frac{\text{Watts}}{\text{kg.}} = W.$	$\frac{W}{f}$.	$B_{max.}$ (corrected).
(2) Frequency $f = 50$ cycles per sec. Secondary turns (voltmeter), $N_3 = 20$.								
10,000	0·586	0·663	1·131	64·0	1·662	1·493	0·02986	9,990
		0·690	1·177	69·3	1·681	1·511	0·03022	
		0·746	1·274	81·2	1·727	1·552	0·03104	
		0·801	1·367	93·5	1·776	1·596	0·03192	
						Sine wave	0·0297	
12,000	0·703	0·803	1·142	65·3	2·436	2·187	0·04374	11,970
		0·860	1·223	74·8	2·480	2·228	0·04456	
		0·973	1·384	95·8	2·590	2·326	0·04652	
		1·063	1·511	114·2	2·680	2·407	0·04814	
						Sine wave	0·0433	
14,000	0·820	0·961	1·171	68·6	3·487	3·131	0·06262	13,900
		1·128	1·375	94·5	3·660	3·289	0·06578	
		1·24	1·511	114·2	3·809	3·421	0·06842	
		1·353	1·650	136·1	3·954	3·552	0·07104	
						Sine wave	0·0617	
15,000	0·879	1·078	1·226	75·2	4·225	3·795	0·0759	14,790
		1·198	1·362	92·8	4·366	3·921	0·07842	
		1·298	1·476	109·0	4·488	4·031	0·08062	
		1·39	1·581	125·0	4·605	4·137	0·08274	
						Sine wave	0·0739	
16,000	0·938	1·19	1·269	80·5	4·798	4·309	0·08618	15,650
		1·283	1·367	93·4	4·912	4·412	0·08824	
		1·335	1·424	101·4	4·991	4·484	0·08968	
		1·395	1·487	110·5	5·065	4·552	0·09104	
						Sine wave	0·0830	
17,000	0·996	1·328	1·334	89·0	5·27	4·73	0·0946	16,480
		1·388	1·394	97·2	5·35	4·81	0·0962	
		1·495	1·501	112·7	5·50	4·94	0·0988	
		1·578	1·584	125·5	5·65	5·08	0·1016	
						Sine wave	0·0897	

TABLE VA.—Correction for Air Flux for Specimen 8A (High-Resistance Material). Correction on $B_{max.}$ for air flux = 4·8H (see Table V).

B (apparent).	H (actual).	H correction on B.	B (corrected).
10,000	3	10	9,990
12,000	6	30	11,970
14,000	21	100	13,900
15,000	43	210	14,790
16,000	73	350	15,650
17,000	108	520	16,480

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Curves connecting W/f and fk^2 were drawn from these results for each value of $B_{max.}$ at which tests were made, those for $B_{max.} = 16{,}000$ being shown in Fig. 4. The slope of each of these curves was measured and is recorded in Table VI. The logarithms of the slopes are plotted against log B for both frequencies in Fig. 5, and it will be seen that

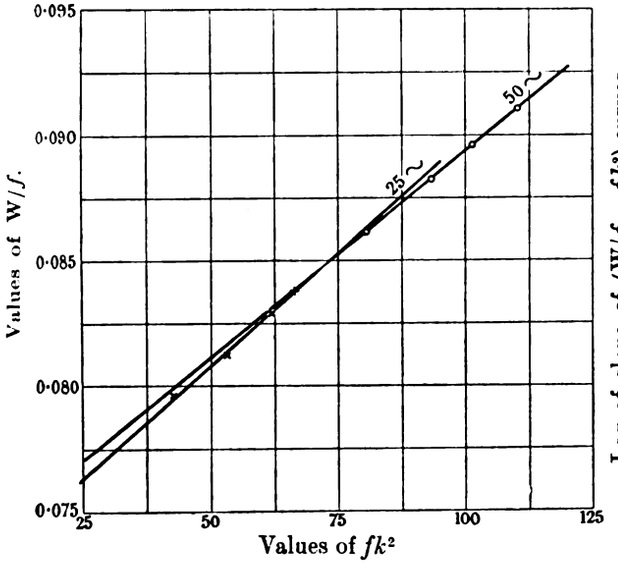

FIG. 4. Curves Connecting W/f and fk^2 for Specimen No. 8A (High-Resistance Material) at B = 16,000.

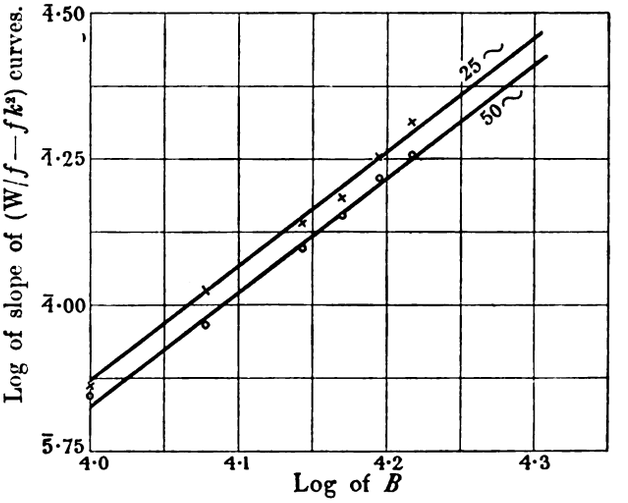

FIG. 5. Logarithmic Curves Connecting Slope of $(W/f - fk^2)$ Curves with B for Specimen No. 8A (High-Resistance Material).

TABLE VI.—Separation of Losses for Specimen No. 8A (High Resistance).

B.	Slope of $\frac{W}{f} - fk^2$ curve $\times 10^4$.		$\frac{W_E}{f}$ (sine wave).	$\frac{W}{f}$ (sine wave).	$\frac{W_H}{f}$.	W_H.	W (sine wave).
	Measured.	From log. curve.					
			Frequency $f = 25$ cycles per sec.				
9,990	0·73	0·75	0·00231	0·02743	0·0251	0·63	0·69
11,970	1·06	1·06	0·0033	0·0398	0·0365	0·91	1·00
13,900	1·38	1·41	0·0043	0·0567	0·0524	1·31	1·42
14,790	1·53	1·59	0·0049	0·0684	0·0635	1·59	1·71
15,650	1·79	1·78	0·0055	0·0774	0·0719	1·80	1·94
16,480	2·11	1·97	0·0061	0·0838	0·0777	1·94	2·10
			Frequency $f = 50$ cycles per sec.				
9,990	0·70	0·67	0·0042	0·0297	0·0255	1·28	1·49
11,970	0·92	0·95	0·0059	0·0433	0·0374	1·87	2·17
13,900	1·25	1·27	0·0078	0·0617	0·0539	2·70	3·09
14,790	1·42	1·43	0·0088	0·0739	0·0651	3·26	3·70
15,650	1·65	1·61	0·0099	0·0830	0·0731	3·66	4·15
16,480	1·81	1·77	0·0109	0·0897	0·0788	3·94	4·49

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the points for each frequency lie very nearly on a straight line the slope of which is slightly less than 2, indicating a practically constant index of B of slightly less than 2. The curve for 50 cycles per second, however, falls appreciably below that for 25 cycles per second, indicating a lower value of the eddy-current coefficient, ξ, at the higher frequency.

Values of the slope corresponding to each value of B have been taken from these curves and are recorded in Table VI. The eddy-current loss for any given form factor may then be found by multiplying the value of the slope by the appropriate value of f^2k^2. For a sine wave $k^2 = 1{\cdot}232$, hence the values of W_E/f for a sine-wave form given in the table are obtained by multiplying the slopes by $1{\cdot}232f$.

The corresponding values of total loss/frequency ($= W/f$) for a sine-wave form are read off the curves connecting W/f and fk^2 by making $fk^2 = 1{\cdot}232f$.

Subtracting W_E/f from W/f, the value of hysteresis loss/f ($= W_H/f$) is found, and hence the hysteresis loss may be obtained. Values of hysteresis loss and total loss in watts per kg. are given in Table VI for each value of B at which tests were made, and for 25 and 50 cycles per second. They are also plotted in Fig. 6, and from these curves

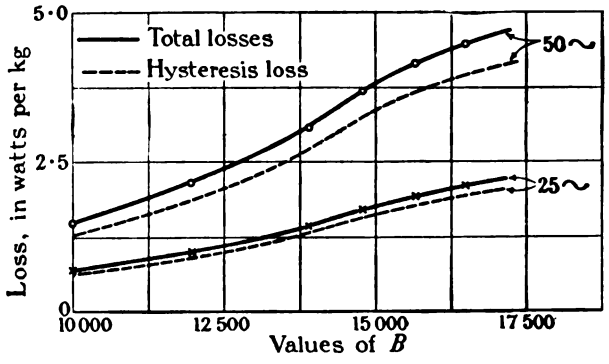

FIG. 6. Curves Connecting Losses and B for Specimen No. 8A (High-Resistance Material).

the final values were read. The results are shown in Table VII, where the hysteresis loss at both frequencies has been expressed in watts per kg. at 50 cycles per second, and where the ballistic values determined from the areas of the static loops, also expressed in watts per kg. at 50 cycles per second, are given for comparison. The exponent of B for the hysteresis loss is also given for each set of results for various ranges of B. Results corresponding to those in Tables VI and VII are given for specimens No. 3A (lohys), No. 6A (special lohys) and No. 12A (medium resistance), respectively, in Tables VIII and IX, X and XI, and XII and XIII.

Discussion of Results and Conclusions.—The results obtained in the more thorough examination of the four representative specimens fully confirm the conclusions drawn from the preliminary general investigation and justify the suggestion made in connexion with it that the exponent for limited ranges of B may reach values of 3·0 or more. In the case of specimen No. 12A the ratio between B = 15,000 and B = 16,000 even exceeds 4·0. It will be seen that the three sets of values of hysteresis loss, which have been

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TABLE VII.—Comparison of Results on Specimen No. 8A (High Resistance).

B.	Total loss in $\frac{\text{watts}}{\text{kg.}}$.		Hysteresis loss in $\frac{\text{watts}}{\text{kg.}}$. At 50 cycles per sec.			Exponent of B for hysteresis loss.		
	At 25 cycles per sec.	At 50 cycles per sec.	From 25-cycle results.	From 50-cycle results.	From static loops.	25 cycles per sec.	50 cycles per sec.	Static.
10,000	0·69	1·49	1·26	1·28	1·26			
						2·02	2·08	1·81
12,000	1·00	2·18	1·82	1·87	1·75			
						2·51	2·52	2·58
14,000	1·45	3·14	2·68	2·76	2·61			
						3·0	2·93	2·73
15,000	1·78	3·83	3·30	3·38	3·15			
						1·93	1·77	2·14
16,000	2·01	4·30	3·72	3·79	3·62			
						1·45	1·27	0·93
17,000	2·19	4·65	4·06	4·09	3·83			

TABLE VIII.—Separation of Losses for Specimen No. 3A (Lohys).

B.	Slope of $\frac{W}{f} - fk^2$ curve $\times 10^4$.		$\frac{W_E}{f}$ (sine wave).	$\frac{W}{f}$ (sine wave).	$\frac{W_H}{f}$.	W_H	W (sine wave).
	Measured.	From log. curve.					
	Frequency $f = 25$ cycles per sec.						
9,990	4·02	3·98	0·0123	0·0490	0·0367	0·92	1·23
11,980	5·76	5·32	0·0164	0·0685	0·0521	1·30	1·71
13,970	6·63	6·84	0·0211	0·0966	0·0755	1·89	2·42
14,960	7·41	7·62	0·0235	0·1146	0·0911	2·28	2·87
15,920	8·14	8·42	0·0259	0·1345	0·1086	2·72	3·36
16,820	9·43	9·21	0·0284	0·1496	0·1212	3·03	3·74
17,700	10·22	10·0	0·0308	0·1618	0·1310	3·28	4·05
	Frequency $f = 50$ cycles per sec.						
9,990	3·64	3·62	0·0223	0·0613	0·0390	1·95	3·07
11,980	5·59	4·88	0·0301	0·0857	0·0556	2·78	4·29
13,970	6·17	6·28	0·0387	0·1194	0·0807	4·04	5·97
14,960	6·61	7·05	0·0434	0·1397	0·0963	4·82	6·99
15,920	8·02	7·82	0·0482	0·1616	0·1134	5·67	8·08
16,820	8·52	8·57	0·0528	0·1816	0·1288	6·44	9·08
17,700	9·19	9·32	0·0574	0·1947	0·1373	6·87	9·74

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TABLE IX.—Comparison of Results on Specimen No. 3A (Lohys).

B.	Total loss in $\frac{\text{watts}}{\text{kg.}}$.		Hysteresis loss in $\frac{\text{watts}}{\text{kg.}}$. At 50 cycles per sec.			Exponent of B for hysteresis loss.		
	At 25 cycles per sec.	At 50 cycles per sec.	From 25-cycle results.	From 50-cycle results.	From static loops.	25 cycles per sec.	50 cycles per sec.	Static.
10,000	1·23	3·08	1·84	1·95	1·73	1·90	1·96	2·00
12,000	1·72	4·31	2·60	2·79	2·49	2·46	2·43	2·30
14,000	2·44	6·01	3·80	4·06	3·55	2·82	2·60	2·57
15,000	2·90	7·04	4·62	4·86	4·24	2·64	2·61	2·73
16,000	3·39	8·19	5·48	5·75	5·06	2·0	2·14	1·92
17,000	3·81	9·24	6·18	6·54	5·68	1·47	1·19	1·78
18,000	4·15	9·94	6·72	7·00	6·29			

TABLE X.—Separation of Losses for Specimen No. 6A (Special Lohys).

B.	Slope of $\frac{W}{f} - fk^2$ curve $\times 10^4$.		$\frac{W_E}{f}$ (sine wave).	$\frac{W}{f}$ (sine wave).	$\frac{W_H}{f}$.	W_H.	W (sine wave).
	Measured.	From log. curve.					
Frequency $f = 25$ cycles per sec.							
9,990	2·81	2·73	0·0084	0·0453	0·0369	0·92	1·13
11,980	3·65	3·67	0·0113	0·0636	0·0523	1·31	1·59
13,950	4·58	4·74	0·0146	0·0906	0·0760	1·90	2·27
14,930	5·12	5·34	0·0165	0·1135	0·0970	2·43	2·84
15,880	5·98	5·93	0·0183	0·1395	0·1212	3·03	3·49
16,770	6·45	6·48	0·0200	0·1575	0·1375	3·44	3·94
17,630	7·02	7·03	0·0217	0·1713	0·1496	3·74	4·28
Frequency $f = 50$ cycles per sec.							
9,990	2·57	2·64	0·0163	0·0533	0·0370	1·85	2·67
11,980	3·55	3·68	0·0227	0·0746	0·0519	2·60	3·73
13,950	4·79	4·65	0·0286	0·1059	0·0773	3·87	5·30
14,930	5·19	5·21	0·0321	0·1322	0·1001	5·01	6·61
15,880	5·91	5·81	0·0358	0·1587	0·1229	6·15	7·94
16,770	6·29	6·36	0·0392	0·180	0·1408	7·04	9·0
17,630	7·39	6·92	0·0426	0·1931	0·1505	7·53	9·66

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TABLE XI.—Comparison of Results on Specimen No. 6A (Special Lohys).

B.	Total losses in watts/kg.		Hysteresis loss in watts/kg. At 50 cycles per sec.			Exponent of B for hysteresis loss.		
	At 25 cycles per sec.	At 50 cycles per sec.	From 25-cycle results.	From 50-cycle results.	From static loops	25 cycles per sec.	50 cycles per sec.	Static.
10,000	1·13	2·67	1·84	1·85	1·77			
						1·94	1·89	1·83
12,000	1·59	3·74	2·62	2·61	2·47			
						2·48	2·65	2·42
14,000	2·30	5·36	3·84	3·93	3·59			
						3·70	3·77	3·90
15,000	2·89	6·71	4·96	5·10	4·70			
						3·15	3·21	3·50
16,000	3·55	8·09	6·18	6·28	5·89			
						2·23	2·25	2·27
17,000	4·03	9·22	7·06	7·19	6·75			
						1·38	1·10	1·52
18,000	4·39	9·83	7·64	7·66	7·36			

TABLE XII.—Separation of Losses for Specimen No. 12A (Medium Resistance).

B.	Slope of $\frac{W}{f}$—fk^2 curve $\times 10^4$.		$\frac{W_E}{f}$ (sine wave).	$\frac{W}{f}$ (sine wave).	$\frac{W_H}{f}$.	W_H.	W. (sine wave).
	Measured.	From log. curve.					
			Frequency f = 25 cycles per sec.				
9,990	1·58	1·57	0·0048	0·0417	0·0369	0·92	1·04
11,980	2·20	2·16	0·0067	0·0579	0·0512	1·28	1·45
13,950	2·76	2·81	0·0087	0·0805	0·0718	1·80	2·01
14,910	3·15	3·16	0·0097	0·0969	0·0872	2·18	2·42
15,840	3·35	3·52	0·0108	0·1230	0·1122	2·81	3·08
16,730	3·87	3·86	0·0119	0·1468	0·1349	3·37	3·67
17,580	3·94	4·23	0·0130	0·1663	0·1533	3·83	4·16
			Frequency f = 50 cycles per sec.				
9,990	1·44	1·42	0·0088	0·0462	0·0374	1·87	2·31
11,980	1·92	2·00	0·0123	0·0640	0·0517	2·59	3·20
13,950	2·69	2·67	0·0164	0·0889	0·0725	3·63	4·45
14,910	3·05	3·06	0·0188	0·1067	0·0879	4·40	5·34
15,840	3·48	3·39	0·0209	0·1341	0·1132	5·66	6·71
16,730	3·94	3·77	0·0232	0·1608	0·1376	6·88	8·04
17,580	3·84	4·15	0·0256	0·1817	0·1561	7·81	9·09

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TABLE XIII.—Comparison of Results on Specimen No. 12A (Medium Resistance).

B.	Total loss in $\frac{\text{watts}}{\text{kg.}}$.		Hysteresis loss in $\frac{\text{watts}}{\text{kg.}}$ at 50 periods per sec.			Exponent of B for hysteresis loss.		
	At 25 cycles per sec.	At 50 cycles per sec.	From 25-cycle results.	From 50-cycle results.	From static loops.	25 cycles per sec.	50 cycles per sec.	Static.
10,000	1·04	2·31	1·84	1·88	1·79			
						1·81	1·78	1·75
12,000	1·46	3·22	2·56	2·60	2·46			
						2·24	2·25	2·27
14,000	2·05	4·49	3·62	3·68	3·49			
						3·08	2·94	3·11
15,000	2·48	5·44	4·48	4·51	4·33			
						4·05	4·13	4·25
16,000	3·20	6·97	5·82	5·89	5·70			
						3·23	3·32	3·33
17,000	3·84	8·40	7·06	7·19	6·96			
						2·32	2·31	2·18
18,000	4·38	9·51	8·06	8·20	7·88			

obtained entirely independently, give exponents which are in very good agreement, the discrepancies between the three sets being apparently due only to incidental errors of measurement and not greater than are to be expected when deriving the exponent for such small ranges of B. Thus, although there are differences between the values of hysteresis loss obtained at different frequencies, probably due to non-homogeneity of the material, it seems certain that the exponents represent fairly accurately the true values for the material. The results thus indicate a very rapid increase of the exponent in the neighbourhood of B = 15,000, followed, as was anticipated, by a rapid decrease to quite low values above B = 17,000. Too much reliance should not be placed on the figures for B = 18,000, but since fairly close agreement was obtained between ballistic and A.C. results it may be assumed that the general shape of the curve, at any rate, follows the course indicated.

So far as the hysteresis loss up to B = 16,000 is concerned, therefore, the results obtained appear to confirm those placed in the third group of the summary at the end of Section II as indicating ratios of variation considerably above 2·0, and although giving somewhat higher values, they are not inconsistent with those in the second group which showed exponents of about 2. With the results in the first group, confirming STEINMETZ'S law up to very high flux densities, however, the observations recorded in the paper appear to be in direct conflict. The most important of these are NICHOLSON'S, which were obtained, as described above, by employing an altogether different method of allowing for distortion of the E.M.F. wave. This renders impossible a detailed

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examination of the points of difference between the two sets of observations, and there seems no obvious explanation of the discrepancy between them.

The significance of the observed departure from STEINMETZ's law may be appreciated by calculating the loss at B = 16,000 from that at B = 10,000, assuming a constant exponent. The observed values at B = 16,000 will be found to be 40–50 per cent. higher than those calculated.

The rapid fall of the exponent at higher flux densities, although as suggested, theoretically probable, does not seem to have been generally observed, and as mentioned previously, the results of BAILY, obtained by the calorimetric method and showing a similar effect, are of considerable interest in this connexion. He found the decrease in the exponent to begin at about B = 18,000. This is a rather higher value than that at which it was observed in the present investigation, but it must be remembered that BAILY's experiments were made about 1895 and were consequently carried out on soft iron sheets instead of on the alloy materials now used, for which the saturation flux density is appreciably lower.

One effect of this successive increase and decrease in the exponent is that, where the losses are determined under such conditions that the flux density varies considerably in different parts of the circuit, as in an actual machine, the exponent of B for hysteresis loss will appear to be much more constant than it really is. This will apply to such results as those recently discussed by E. HUGHES.*

The curves in Figs. 4 and 5, and the results given in Tables VIII, X and XII, show that the slope of the $W/f - fk^2$ curves is slightly less at 50 cycles than at 25 cycles, corresponding to a slightly lower value of the eddy-current coefficient. This holds fairly consistently for all specimens and at all values of B. It is quite a small effect, and like the apparent variation of hysteresis with frequency, is believed to be due to non-homogeneity of the material due to the presence of slight scale.

The slopes of the logarithmic curves plotted from the results in Tables VI, VIII, X and XII, of which Fig. 5 is an example, show that the eddy-current exponent is nearly constant for each specimen and is in general rather less than 2. This tendency at high flux densities has been noted by several previous observers.

To summarise the conclusions deduced from the results recorded, it has been shown that :—

(1) In all the principal types of commercial sheet material in use at the present time, the rate of increase of the hysteresis loss with B is considerably greater at flux densities above B = 10,000 than is indicated by STEINMETZ's law or than appears to have been generally realised. In most specimens over a small range in the neighbourhood of B = 15,000 the variation is as rapid as the 2·5th power of B, and in some cases it exceeds the 3rd or even the 4th power of B.

(2) Above about B = 16,000 the rate of increase falls off very rapidly and drops again to the 1·6th power or even lower.

* E. HUGHES, ' J. Inst. Elec. Eng.,' Vol. 63, p. 35, 1925.

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(3) The eddy-current index at high flux densities is fairly constant and generally rather less than 2.

(4) The apparent values of hysteresis and eddy-current losses in ordinary commercial sheet material depend to a considerable extent on frequency. Experiments not recorded in this paper indicate that this is due to non-homogeneity of the sheet. These effects do not appreciably alter the exponents observed.

In conclusion the author wishes to express his thanks to Messrs. J. Lysaght, Ltd., for kindly supplying the samples of sheet material used in the investigation; to Dr. D. W. Dye, F.R.S., for his help and advice throughout the course of the work; and to Mr. L. H. Ford, B.Sc., for his assistance in carrying out the experiments described.

Appendix I.

Comparison of Permeabilities, Measured Ballistically and by Alternating Current.

Several tests were made in the course of the investigation to compare the values of permeability obtained by A.C. measurements with those given by the ballistic method.

For the A.C. tests the primary of a mutual inductance of 0·01 henry, wound on a marble bobbin, was connected in series with the magnetising winding of the square. Then, after adjusting to any desired $B_{max.}$ by means of the V_{mean} reading in the usual way, the moving-coil galvanometer G (Fig. 1) was connected through the synchronous commutator to the secondary of the mutual inductance, the commutator was again adjusted to give the maximum reading, and the value of V_{mean} indicated was noted. Then

$$V'_{mean} = 4fMI_{max.},$$

where

M = mutual inductance in henrys,

$I_{max.}$ = maximum value of magnetising current in amperes.

f = frequency in cycles per second.

V'_{mean} = reading of moving-coil galvanometer in volts.

From this expression the maximum value of the magnetising current in the square may be at once deduced and, treating the square as a ring in the way described in the paper, a simple calculation gives the value of $H_{max.}$ corresponding to the $B_{max.}$ at which the reading was taken.

Table XIV gives a few typical results obtained on specimen No. 3 with the ballistic values for comparison. These results indicate clearly that at ordinary commercial frequencies the permeability ($B_{max.}/H_{max.}$) measured by alternating current is practically identical with that found ballistically. A considerable number of similar tests were made with the same results.

It seemed worth while to record these results, since N. W. McLachlan,* in a paper on "The Magnetic Behaviour of Iron under Alternating Magnetisation of Sinusoidal Wave-

* N. W. McLachlan, "J. Inst. Elec. Eng.," Vol. 53, p. 809, 1915.

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form,"* published results showing higher values of B, for given values of H, in A.C. than in D.C. (ballistic) tests.

TABLE XIV.

(M = 0·01 henry. Frequency f = 50 cycles per second.)

$B_{max.}$ (apparent).	$B_{max.}$ (corrected).	$V'_{mean.}$	$I_{max.}$	$H_{max.}$	H (ballistic).
10,000	9,990	0·68	0·34	2·74	2·64
12,000	11,980	1·05	0·53	4·27	4·16
14,000	13,970	1·82	0·91	7·32	7·20
15,000	14,960	2·88	1·44	11·6	11·5
16,000	15,920	5·61	2·81	22·6	22·8
17,000	16,820	12·41	6·21	50·0	49·8

This result, of which no theoretical explanation was suggested, and which on theoretical grounds seems highly improbable, was obtained by making use of a much less direct method of measuring the A.C. permeabilities than that described above, involving the use of an oscillograph taking an appreciable current. His results also showed a marked change in the value of $B_{max.}$ for a given $H_{max.}$ when a small change in the wave-shape of the magnetising current was made (form factor of current-wave altered from 1·11 to 1·19), the frequency remaining the same.

This suggests that the effect observed is not due to a genuine difference between the D.C. and A.C. permeabilities, and the author thinks that the more direct measurements recorded above represent with much greater accuracy the true relation between the properties of iron sheet under direct and alternating magnetisation.

APPENDIX II.

Tests on Annealed Strips and Rings (to Eliminate the Effects of Cutting).

In order to determine the magnitude of the effect of cutting and to obtain satisfactorily comparable results, both ballistically and by alternating current, on strips and rings, several complete series of tests were made at flux densities from 10,000 to 18,000 on the medium-resistance specimens Nos. 12A and 12B, both before and after annealing.

The results of A.C. and ballistic tests on specimen No. 12A have already been recorded in Tables XII and XIII, and of ballistic tests on No. 12B in Table IV. A corresponding series of A.C. tests on No. 12B was made at 50 cycles per second by the method described in the paper under " Detailed Examination of Certain Specimens."

Both strips and rings were then annealed together, and the whole series of tests was repeated on both specimens. The complete set of results, with those obtained before annealing, for comparison, is given in Table XV.

* N. W. McLACHLAN, ' J. Inst. Elec. Eng.,' Vol. 53, p. 809, 1915.

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TABLE XV.—Comparison of Values of Hysteresis Loss in Watts per kg. at 50 cycles per second obtained on Specimens 12A and 12B before and after Annealing.

B.	Before annealing.				After annealing.			
	Rings.		Strips.		Rings.		Strips.	
	A.C.	Ballistic.	A.C.	Ballistic.	A.C.	Ballistic.	A.C.	Ballistic.
10,000	2·29	2·11	1·88	1·79	1·72	1·63	1·69	1·66
12,000	3·14	2·92	2·60	2·46	2·42	2·29	2·37	—
14,000	4·19	3·95	3·68	3·49	3·47	3·35	3·39	3·28
15,000	—	4·78	4·51	4·33	4·48	4·40	4·28	4·19
16,000	6·27	6·04	5·89	5·70	5·78	5·54	5·73	5·56
17,000	—	—	7·19	6·96	6·61	6·56	6·85	—
18,000	8·63	—	8·20	7·88	7·26	7·37	7·51	7·42

The results show clearly that the cause of the previous differences between the rings and strips was the greater effect of cutting on the rings. Annealing produced reductions in the hysteresis loss up to 25 per cent. on the rings and up to 10 per cent. on the strips, the reductions naturally being greater at B = 10,000 than at higher flux densities. The resulting values for rings and strips will be seen to be nearly identical. The A.C. and ballistic results are also in very fair agreement, the A.C. values, however, as found in the results embodied in the paper, tending to be rather higher than the ballistic values. In the only case in which this rule is broken—the results at high flux densities on rings after annealing—it is to be noted that the A.C. results are not very reliable, as under these conditions the wave-shape is very distorted and a very large correction has to be made for the form factor.

APPENDIX III.

Accuracy of Ballistic Tests on the Square.

To check the accuracy of results obtained by treating the strips built up with corners in the Lloyd square as a ring for the purpose of making ballistic measurements, permeability curves on specimen No. 3A (lohys) were taken ballistically :—

(1) Treating the specimen as a ring as described in the paper.

(2) Measuring H by flat search-coils placed on either side of the strips near the middle of the magnetising coils, and measuring B by search-coils wound on the same length of specimen as was covered by the H coils.

The results obtained are given in Table XVI.

It will be seen that the discrepancy, except for fairly low values of H, is very small indeed, and that its maximum value is less than 3 per cent.

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TABLE XVI.

H.	B [Method (1)].	B [Method (2)].
2·5	9,670	9,850
5·0	12,580	12,950
7·5	13,930	14,110
10·0	14,580	14,700
15·0	15,210	15,270
25·0	15,800	15,830
50·0	16,710	16,710

A further evidence of the accuracy of the method is given by results quoted in Appendix II for strips and rings annealed together. The close agreement between the ballistic results on the two sets of annealed samples again indicates that no serious error is involved in treating the square as a ring.

APPENDIX IV.

Influence of Scale.

J. D. BALL* and T. SPOONER† have published results suggesting that the increase in the STEINMETZ exponent above B = 10,000 could be explained by the presence of scale on the steel. The scale has a much lower permeability and a much higher hysteresis coefficient than the steel. At low inductions, therefore, the flux is almost entirely carried by the steel and the loss in the scale is negligible. As the induction increases and the permeability of the steel diminishes, more of the flux passes through the scale, which then, owing to its high hysteresis coefficient, adds considerably to the total loss in the sheet. Thus a very rapid increase in the hysteresis loss occurs at high values of B. They found that after the scale had been removed by pickling the exponent was much reduced, though still higher than at low values of B.

Similar tests were made on specimen No. 3A, which was very carefully pickled and the complete tests on it repeated. The final results are given in Table XVII, which is directly comparable with Table IX.

It will be seen from this comparison that the effects of removing the scale are :—

(*a*) To reduce the apparent loss, due, of course, to the removal of the low permeability scale, the more uniform distribution of the flux across the section of the remaining sheets and the consequent lower flux density in the high permeability steel.

(*b*) To reduce to a negligible amount the discrepancies between the ballistic, 25-cycle and 50-cycle results. This suggests that the rather large discrepancies (up to 7·9

* J. D. BALL, 'Trans. Amer. Inst. Elec. Eng.,' Vol. 34, p. 2720, 1915; 'J. Franklin Inst.,' Vol. 181, p. 459, 1916.

† T. SPOONER, 'Electric J.,' Vol. 22, p. 132, 1925.

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per cent.) between ballistic and A.C. values of hysteresis loss are due to the presence of scale and are not inherent in the method of testing. Presumably the flux distribution is different in the ballistic and A.C. tests when scale is present.

(*c*) The maximum values of the exponent of B for the hysteresis loss are not appreciably less than before, but the increase in the exponent takes place at rather higher values of B. This is explained by the actual flux density in the steel being somewhat higher than the apparent value when scale is present.

These results indicate that the increase in the exponent is not a spurious effect due to the presence of scale, but a genuine property of the material, although a larger number of tests on this point would be desirable to confirm the results obtained on specimen 3A.

TABLE XVII.—Comparison of Results on Specimen No. 3A after Pickling.

B.	Total loss in watts/kg.		Hysteresis loss in watts/kg.			Exponent of B for hysteresis loss.		
			At 50 cycles per sec.					
	At 25 cycles per sec.	At 50 cycles per sec.	From 25-cycle results.	From 50-cycle results.	From static loops.	25 cycles per sec.	50 cycles per sec.	Static.
10,000	1·11	2·67	1·68	1·69	1·70			
						1·82	1·74	1·78
12,000	1·52	3·69	2·34	2·32	2·35			
						2·19	2·18	2·06
14,000	2·10	5·09	3·28	3·25	3·23			
						2·58	2·60	
15,000	2·50	5·97	3·92	3·89	—			2·76
						2·81	2·72	
16,000	2·94	6·99	4·70	4·64	4·67			
						1·74	1·77	1·69
17,000	3·30	7·78	5·22	5·16	5·17			
						1·17	1·30	—
18,000	3·58	8·45	5·58	5·56	—			

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Appendix V.

Data from Ballistic Tests.

For many purposes it is of value to know not only the hysteresis loss as given by the total area of the loop, but also the shape of the loop. This may be approximately defined by the values of the maximum H, remanence and coercive force.

The representative samples Nos. 3A, 6A, 8A and 12A were each tested ballistically for hysteresis loops at a number of values of $B_{max.}$ from 10,000 up to 17,000 or 18,000 to give the results recorded respectively in Tables IX, XI, VII and XIII in the columns headed "From static loops." The above-mentioned numerical constants associated with these loops are given in the following Table XVIII.

Table XVIII.—Constants of Hysteresis Loops for Representative Samples.

Specimen No.	$B_{max.}$	$H_{max.}$	Remanence.	Coercive force.	Hysteresis loss in watts per kg. at 50 cycles per sec.
3A (Lohys)	10,000	2·52	8,280	0·86	1·73
	12,500	4·5	10,160	0·92	2·71
	15,000	11·0	11,500	0·99	4·24
	16,000	23·4	11,550	0·99	5·06
	17,000	52·0	11,600	0·99	5·68
	18,000	94·5	11,610	1·0	6·29
6A (Special lohys)	10,000	2·7	7,890	0·89	1·77
	12,000	4·4	9,200	0·96	2·47
	14,000	10·1	10,100	1·0	3·59
	15,000	17·8	10,540	1·01	4·70
	16,000	34·4	10,800	1·01	5·89
	17,000	65·0	10,910	1·02	6·75
	18,000	108·0	10,920	1·02	7·36
8A (High resistance)	10,000	2·86	6,550	0·59	1·26
	12,000	6·11	7,050	0·63	1·75
	14,000	23·0	7,280	0·64	2·61
	15,000	50·0	7,280	0·64	3·15
	16,000	87·0	7,280	0·64	3·62
	17,000	140·0	7,280	0·64	3·83
12A (Medium resistance)	10,000	2·85	7,440	0·91	1·79
	12,000	4·52	8,640	0·98	2·46
	14,000	11·0	9,500	1·02	3·49
	15,000	21·6	9,670	1·03	4·33
	16,000	40·1	9,790	1·03	5·70
	17,000	71·5	9,840	1·04	6·96
	18,000	120	9,860	1·05	7·88

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XV.

ON THE DETERMINATION OF THE DIRECTIONS OF THE FORCES IN WIRELESS WAVES AT THE EARTH'S SURFACE

BY

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ABSTRACT.

A method is described for measuring the inclination of wireless waves to the vertical. The apparatus consists of a Hertzian rod, 30 feet long, with a receiver at the centre. The rod is capable of rotation in a vertical plane about a horizontal axis, which axis can itself be rotated in a horizontal plane, thus allowing the aerial to be capable of rotation in any vertical plane. The axis of rotation is chosen to be perpendicular to the direction of propagation, and the rod is rotated till a minimum is obtained. This position is usually for the rod nearly horizontal, and the angle of deviation from the true horizontal is the angle of inclination of the electric force to the vertical. Measurements were made on various transmitting stations, the wave-lengths varying from 450 metres to 6,900 metres. Various receiving sites in the south of England were used. In most cases the angle of inclination was less than 3°. In certain cases the wave-length of the transmitter varied, and the angle of inclination was greater the shorter the wave-length. The view is expressed that at the conducting surface of the earth the angle of inclination can never be large, no matter what the direction of the incident waves to the surface may be. The actual inclination depends on the conductivity and the frequency of the waves according to the relation, $\sigma = \frac{n}{4 \tan^2 B_0}$, where σ is the conductivity of the earth, n the frequency of the waves, and B_0 the inclination of the electric force to the vertical. Values for the conductivity were obtained by measuring B_0 for a number of different sites in the south of England, these values lying between $0{\cdot}14 \times 10^8$ and $4{\cdot}7 \times 10^8$ (electrostatic units).

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ON THE DETERMINATION OF THE DIRECTIONS OF THE FORCES IN WIRELESS WAVES AT THE EARTH'S SURFACE.

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I.—*Introduction.*

There are two outstanding problems relating to the propagation of wireless waves over the earth's surface which at present remain unsolved, viz.—What is the agency which causes the waves to follow the curvature of the earth, thus rendering long-distance communication possible? And what is the cause of the large and rapid variations of the intensity and apparent direction of the waves, very commonly observed at the receiving station and confined almost entirely to the hours of darkness?

Both phenomena can be explained to some extent by the well-known Heaviside-layer theory, with the modifications proposed by Eccles, but it is generally admitted that further experimental evidence of the existence of the layer is needed. If the theory is correct and is sufficient, it follows that in the case of a receiver experiencing either of the above phenomena, part of the energy must be arriving in a downward direction (*i.e.*, inclined to the horizontal); and that during the occurrence of directional variations, this downcoming wave must have a horizontally polarised component (*i.e.*, with its electric force horizontal). It has, therefore, been generally recognised for some time that a conclusive experimental demonstration of the presence or absence of such waves, by suitable quantitative measurements in three dimensions, would aid considerably in proving or disproving the Heaviside layer theory.

Several workers have carried out investigations on these lines. The most successful of these was T. L. Eckersley,* who obtained results which appear to indicate definitely the presence of an abnormally polarised downcoming wave, and to show that the intensity of this wave is proportional to the magnitude of the directional error. Although it is extremely difficult to find any fault either in the experimental methods adopted or in the analysis based on the results, these conclusions are of such importance that they have naturally been submitted to some criticism, and it is highly desirable to carry out confirmatory experiments.

* T. L. Eckersley, 'Radio Review,' Vol. 2, pp. 60, 231, 1921.

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Other work of this nature has been carried out by AUSTIN,* who attempted to measure the direction of the electric field of an arriving wave by means of a large " Hertzian rod " aerial ; by ERSKINE-MURRAY and ROBINSON,† who employed a combination of Hertzian rod and a frame coil mounted on three axes for determining the direction of propagation of electro-magnetic waves ; by BELLINI,‡ who made observations on the inclination of the magnetic force to the horizontal during the occurrence of directional variations by means of a frame coil rotating about both vertical and horizontal axes ; and by Sir HENRY JACKSON,§ who also used a frame coil with much the same object. As far as can be gathered from the published information, however, it appears that these experiments were accompanied by either instrumental or local errors which prevented the establishment of any definite facts relevant to the general problem of the propagation of waves.

The present investigation consists in following up the line of attack opened by the above workers, in an attempt to obtain more conclusive results. The experimental methods employed are much the same in principle, but a large amount of detailed research has been carried out on the apparatus to improve its accuracy. The present paper describes only the first part of the investigation, the later developments of which are discussed in Papers 16 and 17.||

II.—*Boundary Conditions Imposed at Earth's Surface.*

A preliminary consideration of the problem in hand drew attention to the necessity of taking into account the boundary conditions imposed on the forces of an electro-magnetic wave at a receiver situated on the ground, *i.e.*, effectively at the surface of separation between a non-conducting and a conducting medium. With a perfectly conducting earth, these conditions would imply that at the surface (assumed horizontal) the resultant electric force (E) must always be vertical and the resultant magnetic force (H) always horizontal, whatever might be the inclination of the wave or waves arriving at the receiver. In other words, a wave incident at any angle at the surface of separation will give rise to a reflected wave, the two interfering in such a way as to tend to eliminate the horizontal component of the electric force and the vertical component of the magnetic force.

It was, therefore, clear at the outset of the experiments that if the earth at the receiver were of sufficiently high conductivity it would be useless to attempt to detect down-coming waves by directional measurements at the surface, since the resultant fields

* L. W. AUSTIN, ' J. Wash. Acad. Sci.,' Vol. 40, p. 101, 1921.

† ERSKINE-MURRAY and ROBINSON, British Patent 176,127, 1921.

‡ BELLINI, ' Electrician,' Vol. 89, p. 150, 1922.

§ Sir H. B. JACKSON, ' Wireless World,' Vol. 9, p. 789, 1922.

|| SMITH-ROSE and BARFIELD, ' N.P.L. Collected Researches,' Vol. 21, Papers 16 and 17, pp. 333, 365 (or ' Proc. Roy. Soc.,' A, Vol. 110, p. 580, 1926 ; Vol. 116, p. 682, 1927.

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produced at the surface of the earth would be indistinguishable from those of a horizontally propagated wave. For this reason a knowledge of the value of this conductivity was considered to be necessary as a preliminary to the experiments. Information on the conductivity of various soils is also to be found in tabular form in various text-books (*e.g.*, FLEMING*). These values appear to have been obtained for the most part from the works of BRYLINSKI, LÖWY, SCHMIDT, and ULLER.†

In most of these cases the conductivity measurements were made in the laboratory on artificially treated specimens of soil, and in very few cases were they carried out at radio frequencies. In view of the heterogeneous nature of the surface and subsoil at any ordinary site, and its varying moisture-content, it was not considered that this information was sufficiently reliable and exact for the purpose required. For example, the values given for moist earth—the nearest description to the conditions at Slough for the greater part of the year—vary from 10^7 to 10^9 electrostatic units,‡ limits which are somewhat wide for making satisfactory deductions as to the suitability of the site for the proposed experiments.

Considerable doubt also existed as to what effect, if any, a non-conducting subsoil such as dry sand or rock would have, when, as is usual on level ground in this country, it is covered by either water or soil of sufficient thickness for the maintenance of ordinary vegetation. In these circumstances, therefore, it was decided to determine the conductivity experimentally, employing wireless waves of ordinary commercial frequencies.

III.—*Determination of Conductivity of Ground at Slough.*

The method adopted was to measure the angle of inclination to the vertical, or "forward" tilt, of waves received from various transmitters situated only a few miles away, it being assumed that such waves are vertically polarised§ and propagated horizontally over the earth's surface. This was effected by rotating a straight Hertzian-rod-type receiver about a horizontal axis, placed perpendicular to the direction of travel of the waves, until the signals were reduced to a minimum. The angle B_0 then made by the rod with the horizontal is the angle of inclination of the electric field to the vertical. By a simple modification of ZENNECK's formula for the ratio of the horizontal to the vertical component of the electric field of such a wave, of frequency "n," the conductivity (σ) in electrostatic units is given by the expression

$$\sigma = \frac{n}{4 \tan^2 B_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

* J. A. FLEMING, 'Principles of Electric Wave Telegraphy,' 4th Ed., p. 623 (Longmans, London, 1919).

† BRYLINSKI, 'Bull. Soc. Internat. Elec.,' Vol. 6, p. 255, 1906; LÖWY, 'Ann. der Phys.,' Vol. 36, p. 125, 1911; H. SCHMIDT, 'Jahrbuch Drahtlosen Telegraphie,' Vol. 4, p. 636, 1911; K. ULLER, *ibid.*, Vol. 4, p. 638, 1911.

‡ To convert to electromagnetic units, divide by v^2 ($= 9 \times 10^{20}$ approx.).

§ The term "vertically polarised" implies that the electric force is contained in the vertical plane of propagation.

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to a close degree of approximation, when B_0 is reasonably small and σ is large compared with n; both of these conditions were found to be valid in the experiments under consideration.

The Hertzian rod used at Slough consisted of a copper wire, 30 feet long, stretched on a light wooden beam in the manner of a bowstring, the centre of this beam being carried on a horizontal axis mounted at the top of a wooden tripod 20 feet high. Leads from the centre of the rod were taken to an amplifying receiver contained in a screened box mounted on a centre turntable, which, with the horizontal axis supporting the rod, was capable of complete rotation about a vertical axis. By this means the rod-aerial could be rotated in any required vertical plane. The success and accuracy of the instrument are largely due to the efficiency of the screening of all the apparatus and leads except the wire forming the aerial, and in the present arrangement this is so satisfactory that, when using a nine-valve amplifying receiver, the direct pick-up of signal from any transmitting station is negligible.

In making an observation the horizontal axis of rotation of the rod is set at right angles to the direction of the transmitting station, and the rod is rotated in its own vertical plane until the received signal is reduced to a minimum. The angle then made by the rod with the horizontal is the angle of inclination (B_0) of the electric field with the vertical. The minima are usually not quite sharp, owing to the fact (shown theoretically by ZENNECK) that the electric force is elliptically polarised in the plane of propagation of the wave. In each position the observation is repeated, after reversal of the leads between rod and receiver, and the procedure is then repeated with the horizontal axis turned through 180°. This process eliminates the error which would otherwise be caused if any residual direct pick-up of signal were still present, and also that due to any incorrect setting of the scale. The maximum error of the individual observations made is considered to be about a degree, although the mean of a number of readings is probably of a higher order of accuracy.

A series of measurements was made on wave-lengths varying from 450 metres to 6·9 km., and the values of B_0, together with the conductivity values (σ) calculated therefrom, are given in Table I.

In the first three cases in Table I it is seen that the quantity measured (B_0) is of the same order of magnitude as the limits of accuracy of the apparatus, and it was therefore not considered justifiable to use these results for the calculation of conductivity.

From these observations it will be seen that the mean value of the effective conductivity of the ground at Slough for wireless-wave propagation was found to be about $0{\cdot}7 \times 10^8$ electrostatic units. This mean value obtained undoubtedly represents a much closer approximation to the actual value than could be obtained from the previously existing information referred to above.

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TABLE I.—Values of Forward Inclination of Electric Force measured at Slough for Calculation of Conductivity of Earth.

Transmitting station.	Distance (miles).	Wave-length (km.).	Frequency (kc. per sec.).	Angle B_0 (deg.).	Conductivity σ (E.S.U.).
Northolt	9·0	6·90	43·5	0·7	
Ongar	36·0	3·80	79·0	0·7	
Teddington	11·5	2·60	116·0	0·6	
Teddington	11·5	0·75	400·0	2·1	$0·7 \times 10^8$
Teddington	11·5	0·75*	400·0	2·0	$0·8 \times 10^8$
Teddington	11·5	0·45*	670·0	2·7	$0·7 \times 10^8$
		Mean value of conductivity			$0·7 \times 10^8$

* Indicates spark transmission ; all others continuous wave.

IV.—*Boundary Conditions at Slough.*

Having gained the necessary knowledge of the conductivity of the ground at the site proposed for the experiments, it was now possible to examine the boundary conditions and calculate exactly what would be the direction of the resultant fields produced by a down-coming wave, if it existed. It will first be assumed for simplicity that a single wave only is arriving at the receiver, travelling in a downward direction with an angle of incidence θ to the normal. Referring to the resultant electric and magnetic forces E and H, due to the combination of the downcoming incident wave and the up-going reflected wave :—

Let B be the inclination of E to the vertical in the plane of propagation of the wave.

Let A be the inclination of E to the vertical in the plane at right angles to the plane of propagation.

Let D be the inclination of H to the horizontal plane.

Then it can be shown from the electromagnetic equations for reflection at a conducting surface that,

(*a*) for a vertically polarised wave (*i.e.*, E in plane of incidence and, therefore, $A = D = 0$),

$$\tan B = \tfrac{1}{2}\sqrt{\frac{n}{\sigma}} \times \frac{1}{\sin\theta}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

and hence from (1)

$$\tan B = \frac{\tan B_0}{\sin\theta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

(*b*) for a horizontally polarised wave (*i.e.*, E is perpendicular to plane of incidence, and, therefore, $A = \frac{1}{2}\pi$, $B = 0$),

$$\tan D = \tfrac{1}{2}\sqrt{\frac{n}{\sigma}} \times \sin\theta \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$= \tan B_0 \times \sin\theta. \quad \ldots\ldots\ldots\ldots \quad (5)$$

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These formulæ are approximations which are substantially true if σ/n is large compared with K (the dielectric constant of the earth). Both the electric field in case (*a*) and the magnetic field in (*b*) are slightly elliptical in form, and the angles given by equations (2) and (4) give the directions of the major axes of these ellipses.

Dealing first with equation (5) it is seen that the value of the angle D increases with the angle of incidence and is greatest for grazing incidence ($\theta = 90°$) when we have $D_{max.} = B_0$. Now the experiments already described have shown that at Slough the angle B_0, and hence $D_{max.}$, varies from a small amount with long waves to about 2·5° at 450 metres, and by extrapolating to a wave-length of 100 metres, the value of B_0 and $D_{max.}$ becomes 4·6°. If equation (3) is now referred to it will be seen that tan B is inversely proportional to sin θ. Now, for all practical conditions θ is probably never less than 45°, and the maximum value of B therefore becomes $B_{max.} = \sqrt{2}\, B_0$, and is therefore only 1·4 times the value of $D_{max.}$ above. Since, as has already been shown, B_0 does not exceed 3° at Slough on commercial wave-lengths, $B_{max.}$ would be very difficult to distinguish from B_0 with the apparatus used up to the present.

A prediction of the values of the angles in the case when two or more waves are arriving simultaneously at different angles of incidence is naturally quite impossible unless the relative amplitudes and phases of the individual waves are known. A little consideration shows, however, that the tendency will generally be for the inclination to be *less than* those obtained for single waves. The exceptions to this occur when the conditions are such that the field due to the down-coming wave and that due to the direct wave are nearly opposite in phase, in which case they will almost neutralise each other, and the resultant field will therefore probably be too small to detect. Furthermore, although in the case of a single abnormally-polarised incident wave we have A = 90°, the resultant electric force at the surface is very small in magnitude if σ/n is large, owing to the almost complete interference between the incident and reflected waves. If, therefore, a direct normally polarised wave of any appreciable magnitude is present, the resultant electric force is inclined sideways at an angle A, which can be shown to be not greater than D.

This brief analysis, therefore, leads to the conclusion that whatever may be the causes of the phenomena of long-distance transmission and directional errors as observed at Slough, they could not be expected to give rise to resultant fields at the receiver differing sufficiently in direction from those of the direct, horizontally propagated wave to be distinguishable with the apparatus available. For further progress it therefore seemed necessary either,

(*a*) to improve the accuracy of the apparatus or to use another method to enable inclinations of the desired magnitude to be detected;

(*b*) to carry out the experiments on short wave-lengths of 100 metres or less; or

(*c*) to find an alternative site at which the ground is of much lower conductivity than at Slough.

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The last course was chosen as the first step, since from the published information on the resistance of the earth (already referred to) it appeared to be quite possible to find a site at which the conductivity would be as low as 10^6, on which the values of B and D might be expected to exceed 15°, and therefore might be conveniently measurable. It was therefore decided to measure the conductivity of a number of other kinds of soil in different parts of England in an attempt to find a more suitable site.

V.—*Measurements of Conductivity of Different Soils in Southern England.*

For this purpose a portable Hertzian-rod receiver was constructed, very similar in design to that installed at Slough, as described in Section III above, but on a smaller scale. The straight wire forming the aerial was only 15 feet long and the height of the supporting tripod only 5 feet. Owing to the reduction in size of the aerial system, the relative magnitude of the energy directly picked up by the tuning circuits, amplifiers, subsidiary apparatus and connecting leads, was increased. By careful detailed attention to the screening of the apparatus, however, this difficulty was overcome, and the apparatus was found to be quite as accurate as the larger instrument; the instrumental error was certainly less than 1°, and the mean of several readings, as usually taken, was probably correct to about 0·5°.

With this apparatus experiments were carried out on some ten different sites, on subsoils of as varied a nature as could be found in the south of England—viz., clay, sand, chalk, granite, and serpentine rock. At several of the sites the measurements were repeated on a second occasion in an attempt to detect possible variations due to change in season or other causes. The results of these measurements, carried out from May to September, 1924, are summarised in Table II.

This table shows that the effective conductivity of the ground at these various sites lies between $0{\cdot}14 \times 10^8$ and $4{\cdot}7 \times 10^8$, and the state of the surface of the ground does not appreciably modify this value. Omitting, in the first place, the results obtained in Cornwall, it is seen that the conductivities are of the same order as that at Slough, which gives an average representative value for the various sites selected well away from the coast. The actual value of σ at Slough, given in Table II, is somewhat higher than that obtained in the earlier measurements, given in Table I.

This discrepancy was discovered later to be due to the fact that the inclination of the electric force to the vertical varied in different parts of the field, as shown by the portable apparatus, and is evidently due to some local cause (*e.g.*, the masses of trees in the vicinity) such as has been previously shown to give rise to errors in the apparent bearings of certain transmitting stations. This variation is not, however, large enough to affect any of the conclusions which have been drawn.

For comparison purposes, it is interesting to note that the previously published values of the conductivities range from 10^9 to 10^6 or less for specimens of moist and dry earth, sand, or clay. The more uniform values obtained in the present measurements

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TABLE II.—Results of Measurements on Forward Inclination of Electric Force (Angle B_0) at various Places with a Portable Rod Receiver.

Site.	Nature of Subsoil.	State of surface.	Date of test.	Value of B_0 measured on			Distance in miles.		Remarks.	Average value of σ obtained from B_0.†
				2LO, 365 m.	5HW, 450 m.	5HW, 750 m.	From 2LO.	From 5HW.		
				Deg.	Deg.	Deg.				
Stoke Common, Bucks	Clay	Very wet	8.5.24	3·1	2·1	1·3	20	14	—	$1·3 \times 10^8$
Walton Heath, Surrey	Sand	Moist	22.5.24	3·8	2·8	2·4	17	10	—	$0·6 \times 10^8$
,, ,, ,,	,,	Very dry	8.7.24	3·3	2·8	2·1	17	10	After four weeks hot, dry weather	$0·7 \times 10^8$
Chesham Bois, Bucks	Chalk	Wet	5.6.24	1·6	1·2	0·7	25	23	After 24 hours continuous rain	$4·7 \times 10^8$
,, ,, ,,	,,	Very dry	10.7.24	1·5	1·5	0·9	25	23	After four weeks hot, dry weather	$3·2 \times 10^8$
Marlborough Downs, Chisledon, Wilts	,,	,,	1.7.24	—	1·2	0·9	68	60	Measurements confirmed on longer waves, 2,600 m. to 6,900 m.—B_0 diminishing to 0·4 on latter wave	$4·0 \times 10^8$
Slough, Bucks ...	Clay	Moist	3.7.24	2·4	1·8	1·3	18	11·5	—	$1·5 \times 10^8$
,, ,, ...	,,	Dry	7.7.24	2·0	1·7	1·1	18	11·5	—	$2·1 \times 10^8$
Princes Risborough, Bucks	Chalk	,,	15.7.24	3·5	3·6	0·9	33	29	Taken on sloping ground	$1·6 \times 10^8$
,, ,, ,,	,,	Moist	5.8.24	1·7	1·5	1·4	33	29	On level site	$2·2 \times 10^8$
The Sands, Farnham, Surrey	Sand	Dry	17.7.24	2·1	2·1	1·4	33	25	—	$1·5 \times 10^8$
Teddington, Middlesex	,,	,,	24.7.24	1·7	1·4	0·7	11	1·0	—	$4·2 \times 10^8$
				5WA, 351 m.	6BM, 385 m.	FFU and GLD, 600 m.	From 5WA.	From 6BM.	From GLD.	
The Lizard, Cornwall	Serpentine	,,	9.9.24	1·9	1·5	—	134	152	23	On barren ground with outcropping rock; 1½ miles from coast	$2·4 \times 10^8$
,, ,, ,,	,,	,,	10.9.24*	5·6	5·9	1·9	134	152	22	On rocky cliff path, ¼ mile from coast	$0·51 \times 10^8$
Land's End, Cornwall	Granite	,,	11.9.24*	7·1	—	5·3	143	169	1·0	On rocky cliffs, 200 yards from coast	$0·17 \times 10^8$
,, ,, ,,	,,	,,	11.9.24*	5·7	7·4	6·7	143	169	1·0	As above; 100 feet nearer coast	$0·14 \times 10^8$

* *Note.*—The ground at the sites chosen at the Lizard and Land's End on September 10 and 11 was of very broken surface and varied nature. This, together with the proximity to the cliffs, probably accounts for the relatively large discrepancies in some of the results. In each case it was noticed that a slight shift in position of the apparatus resulted in a change in the values obtained, and in several positions the electric and magnetic fields were subject to appreciable sideways inclinations.

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are probably due to the fact that for wireless wave propagation the effective conductivity is not solely determined by the condition at the immediate surface. In England, even in the hottest summer weather, it is doubtful if the soil ever becomes thoroughly dry to a depth of more than a few feet, and it is the moist soil below which will in this case determine the forward inclination of the wave-front. Also, as HACK* has previously shown, the presence of a layer of ground water at a depth of a fraction of a wave-length in a poorly conducting soil will almost entirely annul the forward tilt of the wave-front which would otherwise be set up at the surface.

Referring now to the measurements made in Cornwall, as recorded at the end of Table II, it is seen that in the first of these, at the Lizard, the conductivity obtained is similar to that at Slough. This is undoubtedly explained by the fact that the highly insulating serpentine rock subsoil was mostly covered by a layer of one or two feet of surface soil with a moderate moisture-content, and that only at intervals over the surface was the rock projecting through the upper soil. As the coast is approached the depth of the soil layer decreases, and within a quarter of a mile of the edge of the cliffs the measurements show an appreciable drop in the conductivity values. For these latter measurements it was extremely difficult to find suitable sites, and the discrepancies occurring in the values of B_0 measured on the different wave-lengths are considered to be due to the uneven rock surface, the consequently varying depth of surface soil, the stratification of the rocks, and the trapping of rain-water in pools on the rock surface below the soil. In any case the sites of these measurements would not be suitable for the proposed experiments on the reception of downcoming waves, since for the study of wave-reflection problems a uniform surface of dimensions greater than a wave-length is necessary, and these particular sites were less than a wave-length from cliffs of two or three hundred feet in height. It is possible that at a more satisfactory site further inland lower conductivity values would be obtained after a hot, dry summer, long enough to dry out the soil down to the rock surface ; but the conductivity at such a place would, naturally, be subject to a large seasonal variation.

It is, therefore, to be concluded from these experiments that, although at some of the places investigated the conductivity is slightly lower than that at Slough, the difference is not sufficiently great to justify the transfer of any of the experimental work. The difference in the maximum possible inclinations of the electric and magnetic forces as obtained from the theory already given is, in fact, almost inappreciable at all the sites investigated.

While not strictly relevant to the problem in hand, it is interesting to record here that during these measurements justification was obtained for the assumption that the inclination of the wave-front was related to the normal to the ground surface at the receiver, whereas the angles measured on the apparatus were related to the true vertical. At most places a suitable horizontal site could be obtained, but it was sometimes necessary to work on slightly sloping ground and make the necessary corrections for the direction

* J. ZENNECK, ' Wireless Telegraphy ' (English Translation), p. 260. McGraw Hill, New York, 1915.

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and magnitude of the slope (usually less than 1°). On one occasion, however, a site was intentionally selected on the side of a hill sloping at about 16°, but after correcting for this, it was found that the values of B_0 agreed with those measured on a level site in the same neighbourhood.

VI.—*Confirmatory Experiments at Slough.*

In view of the conclusion arrived at in the last section, it is of sufficient interest to record the result of a series of measurements of the directions of the electric and magnetic forces made at Slough, mainly with the object of confirming experimentally the deductions made from the boundary conditions considered in Section IV.

(*a*) *Method of Measurement.*—The direction of the electric field was measured by means of the larger Hertzian-rod apparatus described in Section III. The direction of the magnetic field was measured by means of a frame coil capable of rotation about both vertical and horizontal axes. With its plane vertical, the coil is rotated about the vertical axis to the position of minimum signal and the ordinary wireless bearing is observed. The plane of the coil in this position is then parallel to the horizontal component of the magnetic field of the arriving wave. The coil is now turned through 90° about its vertical axis to the maximum signal position, and then rotated about its horizontal axis until a second position of minimum signal is obtained, when the angle made by the plane of the coil with the horizontal is observed. This reading gives the elevation of the resultant magnetic field due to the arriving wave, and the direction of the field is the line of intersection of the planes of the coil when in the two minimum positions above determined. In the complete measurements the observations are always repeated after reversing the connexions of the frame coil, and the whole operation is then repeated after rotating the coil through 180° about a vertical axis. The mean readings so obtained are practically free from any scale errors, and also from the effects of any residual direct pick-up on the leads and coupling coils of the receiver.

The auxiliary apparatus used with this frame coil is the same as that employed with the Hertzian-rod receiver. In addition to the screening of the tuning circuits and amplifiers in a metal box, the operator and whole of the apparatus are contained in a wooden hut which is well insulated from earth and entirely surrounded by a wire screen comprising open untuned loops. This type of screen has been shown previously* to eliminate antenna effect on the frame-coil system. The accuracy of the observations obtainable with this apparatus was considered to be about the same as that of the Hertzian rod instrument.

(*b*) *Discussion of Results.*—Table III gives a summary of the results of a series of systematic measurements of the directions of the electric and magnetic fields as observed simultaneously at Slough, over a period of seven months, on various wave-lengths.

* R. H. Barfield, ' J. Inst. Elec. Eng.,' Vol. 62, p. 249, 1924.

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In this table, A, B, B_0 and D are the same quantities defined in Section IV, p. 323; and C is the angle between the observed direction of the horizontal component of the magnetic field and its normal direction, *i.e.*, C is the error in the observed wireless bearing.

TABLE III.—Summary of Results of Simultaneous Directional Measurements of Electric and Magnetic Fields at Slough.

Transmitting station.	Distance.	Wave-length.	Frequency.	Extreme readings over a period of seven months.				B_0 calculated from $\sigma = 10^8$.
				A.	B.	C.	D.	
	miles.	km.	kc. per sec.	deg.	deg.	deg.	deg.	deg.
Nantes	302	9·00	33·3	+0·1 −1·3	0·0 −0·5	−0·9 −1·7	−0·3 −0·7	+0·5
Leafield	48·5	8·70	34·5	+1·1 −0·7	+0·5 −1·3	+12·0 −11·2	+1·0 −0·7	+0·5
Tours	165	6·80	44·2	0·0 −1·0	+1·1 0·0	+0·6 −1·4	+0·3 −1·0	+0·5
Königswusterhausen ...	608	5·30	56·6	0·0 −0·3	+1·5 0·0	+3·1 −9·7	+1·0 −0·3	+0·6
Nauen	576	4·70	63·8	+0·5 −1·0	+1·5 +1·0	+9·8 −5·7	+0·7 0·0	+0·6
Ongar (GLO)	36	4·40	68·2	+0·5 −0·5	+1·5 −0·7	+4·2 −6·3	+0·8 −0·3	+0·8
Ongar (GLB)	36	3·80	79·0	+1·0 −1·0	+1·5 0·0	+2·3 −2·5	+1·3 0·0	+0·8
Paris (FL)	221	2·60	116·0	0·0 −0·5	+1·0	+0·1	0·0 −0·1	+0·9
Teddington	11·5	0·75	400·0	−0·1	+2·1 +2·0	+0·6 +0·4	+0·1 −0·5	+1·7
Teddington	11·5	0·45	670·0	−0·3	+2·7 +2·6	+1·2 +0·5	+0·1 −0·1	+2·2

Many of the observations summarised above were purposely made on stations and at periods which experience had shown to be most likely to produce variations in C (*i.e.*, directional errors or night effect) and it will be seen that variations as large as 23° were obtained. Yet the results show that on these occasions there were no corresponding variations in the values of A, B and D. In fact, throughout the whole series of observations and over the entire range of wave-lengths the values of A and D do not depart from zero by more than 1·3°, a value which is inappreciably greater than the limit of accuracy of the apparatus for the individual readings. It was frequently noticed during the observations on the frame coil apparatus that the minimum obtained with the coil vertical was blurred, a common occurrence at night-time on wireless direction-finders. On all occasions, however, the minimum with coil in the horizontal position was sharp

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and well-defined, thus indicating that the magnetic field was of elliptical form in a horizontal plane only.

Again referring to Table III, many of the stations are sufficiently distant for the phenomena of long-distance transmission to enter, and according to the Heaviside layer theory most of the energy would be coming downwards from the layer. Yet in no case do the values of B differ by an appreciable amount from the theoretical values of B_0 as recorded in the last column of the table, these values of B_0 having been calculated from the conductivity of the ground at Slough as found experimentally. These experiments are thus seen to support the conclusions drawn from the theoretical considerations in Section IV, and cannot, therefore, be considered as evidence for or against the existence of downcoming waves, as required by the Heaviside layer theory.

VII.—*Summary.*

This paper describes the commencement of a theoretical and experimental investigation into the determination of the directions of forces in electromagnetic waves, particularly with a view to elucidating many problems connected with the propagation of wireless waves over the earth's surface. So far as it has gone the investigation has drawn attention to the fact that the success of experiments to detect downcoming waves by directional measurements depends to a large extent upon the conductivity of the ground on which the experiments are carried out; for the boundary conditions imposed by this conductivity are such that the resultant fields at the surface tend to have the same direction in space whatever the polarisation and inclination of the component waves. In the case of a perfectly conducting earth, for example, the electric force will always be vertical and the magnetic force horizontal at the earth's surface.

It is concluded from the measurements made that the ground at Slough has an effective conductivity of about 10^8 (in absolute electrostatic units), and using this value to determine the boundary conditions, it is found that on commercial wave-lengths the resultant fields of downcoming waves, incident at any likely angle (*i.e.*, greater than 45° to the vertical), will not be inclined sufficiently at the surface to distinguish them from horizontally propagated waves.

In a search for a site of low conductivity on which experiments might be carried out with more success, an extension of the measurements was made to various parts of Southern England, which showed that all typical soils possessed conductivities of the same order as at Slough, varying between the limits of $0{\cdot}6 \times 10^8$ and $4{\cdot}7 \times 10^8$. Only in the extreme cases of very rocky and broken surfaces was a conductivity as low as $0{\cdot}14 \times 10^8$ experienced, and in these cases the very limited extent of suitable ground rendered the site impracticable for its required purpose. It is therefore concluded that in England, at any rate, there is no more favourable site than Slough, as regards the earth's conductivity, at which to continue the general investigation.

Finally, a summary is given of the results of measurements at Slough of the directions

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of the electric and magnetic fields in arriving waves, over a period of seven months, by day and night and over various distances; these results support the above conclusions as to the impossibility of detecting inclined fields at Slough, even during the occurrence of directional errors and long-distance reception at night.

The best methods of continuing the investigation in the light of these conclusions are mentioned in Section IV. It appears that in order to obtain positive results with this directional method of attacking the propagation problem, it will be necessary to :—

(1) Improve the accuracy of the apparatus and then repeat the experiments on the same site; or

(2) Repeat the experiments with the existing apparatus, but on ground of lower conductivity; or

(3) Repeat the experiments on the same site and with the same apparatus, but on a shorter wave-length.

All these are possible lines of advance, though all have their difficulties, and on the whole a combination of them is probably the best course to take.

It appears that in order to find a site of sufficiently low conductivity for the purpose, it would be necessary to go to some part of the world which is much drier than England. By choosing some locality with a very limited rainfall it should be possible to find a site where the surface water is sufficiently low down not to affect the electromagnetic fields at the surface. On such a site it should be possible to obtain conclusive results with apparatus of the existing design.

This work was carried out for the Radio Research Board of the Department of Scientific and Industrial Research.

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XVI.

AN INVESTIGATION OF WIRELESS WAVES ARRIVING FROM THE UPPER ATMOSPHERE

BY

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ABSTRACT.

Continuing the work described in Paper 15 the authors have developed modified apparatus for use on shorter wave-lengths. Operating on wave-lengths in the neighbourhood of 400 m., it has been found that the directions of the component forces in the arriving waves can vary considerably from their normal daytime values. Such variations begin at times in the proximity of sunset or after total darkness prevails, and accompany the more commonly known variations in signal strength and apparent bearings as observed on a direction-finder. The magnitudes of the variations already observed on the two independent sets of apparatus provide direct evidence of the reception of wireless waves arriving at the earth's surface at a comparatively small angle of incidence (*i.e.*, 30° or less to the vertical). It also appears from the observations that the downcoming waves can be polarised with the electric force in either a vertical or horizontal plane. By combining the above measurements with simultaneous observations of the strength of the vertical electric force of the arriving waves, a means is provided of determining the angle of incidence of both the vertically and horizontally polarised components of the downcoming waves. Typical results of this kind, carried out at Slough on the transmissions from Bournemouth, are given. These show that the angles of incidence can have any value between the limits of 16° and 34°. If the latter value represents the case of a simple wave which has been symmetrically reflected or deflected from the upper atmosphere, the maximum height of the path is found to be 55 miles or 88 km. This is in good agreement with recent estimates for wave-lengths of the order employed. The results obtained also furnish data by which the relative intensities of the downcoming and direct waves may be determined. Some measurements have given values varying from 0·3 to 1·4 for this ratio, showing that under the conditions of these experiments the intensity of the downcoming wave can be of the same order as that of the direct wave. By substituting the known value of the earth's conductivity in the theoretical formula given by SOMMERFELD, a value of the attenuation of the waves travelling from Bournemouth to Slough can be obtained. On such a basis, the " reflection coefficient " of the upper layer is found to be about 0·2.

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AN INVESTIGATION OF WIRELESS WAVES ARRIVING FROM THE UPPER ATMOSPHERE.

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[Reprinted from the 'Proceedings of the Royal Society,' Series A, Vol. 110, 1926.]

Part I.—The Directions of the Forces in Wireless Waves at the Earth's Surface.

(1).—*Preceding Paper and Continuation of the Work.*

In the preceding paper* the authors drew attention to the desirability of obtaining conclusive experimental evidence as to the actual path of travel of wireless waves between the transmitter and receiver. According to modern theory the upper portion of the earth's atmosphere acts in the nature of a deflecting medium, from which some or all of the energy reaching it from the transmitter is returned to the earth's surface, these downcoming waves having a variable plane of polarisation. It was thought to be possible to demonstrate the presence or absence of such downcoming waves by making measurements at the earth's surface of the directions of the forces in the resultant waves received. The method adopted for the determination of the direction of the electric force in waves from near-by transmitters gave a means of measuring the effective conductivity of the earth's surface for wireless frequencies. This conductivity was found to be of such a high value that on theoretical grounds it was shown that the possibility of distinguishing between downcoming waves and horizontally propagated waves by directional measurements of the resultant forces would be very difficult on the medium and longer wave-lengths employed in commercial wireless communication. This deduction was confirmed by measurements carried out during 1924 at the Radio Research Station of the Department of Scientific and Industrial Research. Operating on transmissions of wave-lengths ranging from 2,600 to 9,000 metres, and under conditions where it was known that the usual phenomena (including night effects) of long-distance wireless transmission were present, the results showed that the directions of the resultant electric and magnetic forces in the received waves provided no definite evidence for or against the existence of downcoming waves.

The apparatus originally set up at the Radio Research Station, Slough, has been in constant use from October, 1924, until December, 1925, for similar measurements.

* Smith-Rose and Barfield, 'N.P.L. Collected Researches,' Vol. 21, Paper 15, p. 317; or 'Proc. Roy. Soc.,' A, Vol. 107, p. 587, 1925.

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The transmissions from Leafield on 12,400 metres and Nantes on 9,000 metres have been kept under constant observation, and occasional experiments have been carried out on other transmitting stations using wave-lengths about 5,000 metres. The results obtained have fully confirmed the deductions previously made.

(2).—*Reconstruction of the Apparatus for Shorter Waves.*

Since the theory indicated that changes in direction of the resultant forces would be more easily detectable on shorter wave-lengths, it was decided to repeat the measurements under these conditions, and, owing to the presence of several very suitable and continuous transmissions from broadcasting stations, the wave-length band of 300 to 500 metres was selected. Owing to the fact that the power of these transmitting stations is very much less than those operating on the longer wave-lengths, an increase in the sensitivity of the receivers was necessitated. This, together with the need for overcoming various instrumental errors of increased importance at shorter wave-lengths, led to the construction of a new rotating coil receiver and to some modifications of the Hertzian rod receiver. The improvements effected are, however, only in matters of detail, and the principle of operation of both sets of apparatus remains unaltered.

(3).—*Typical Results obtained with Rotating Rod and Coil.*

(*a*) *Electric Force.*—In the use of the Hertzian rod receiver the same procedure was adopted as formerly, viz. : to determine first the angle (A) by which the electric force departs from the vertical in a vertical plane perpendicular to the direction of travel, and second, the angle (B) by which the electric force departs from the vertical in the vertical plane of propagation of the wave. When taking observations in this manner in the day time, it is found in the first position that the signal minimum is quite sharp and that the angle A can be determined to within a fraction of a degree ; and also that consecutive observations taken over a period of one or two hours give constant values of A, a maximum variation of less than 2° having been observed during several months' observations on three stations operating on wave-lengths of about 400 metres. In the second case, while the angle B can be measured to an accuracy of the order of 1°, the signal minimum is found to be not silent, but blurred to such an extent that it is necessary to swing the rod through about 4° to 8° in making a determination. Within the limits of accuracy of the apparatus, the angle B is also found to be constant in the day time. Table I gives the values of A_0 and B_0, which are the mean day-time values of these angles as measured on three transmitting stations over a period of several months.

In the simplest case, for a wave travelling horizontally along the ground, the value of A_0 should be zero, and B_0 should have a positive value equal to the forward tilt of the wave as determined by the conductivity of the ground and the frequency of the wave.

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TABLE I.—Values of Normal Daylight Tilts of Electric Force.

Transmitting station.	Wave-length (metres).	A_0 (degrees).	B_0 (degrees).	Value of ground conductivity (σ) calculated from B_0. (E.S.U.)
Cardiff	353	+ 0·1	+ 3·0	$0{\cdot}79 \times 10^8$
London	365	+ 0·2	+ 2·7	$0{\cdot}95 \times 10^8$
Bournemouth	386	+ 0·9	+ 2·8	$0{\cdot}81 \times 10^8$
Birmingham	479	− 0·1	+ 2·3	$0{\cdot}97 \times 10^8$

From the values of B_0, the conductivity of the ground at wireless frequencies can be calculated, and the results are given in column 5 of Table I. The mean conductivity so obtained, for wave-lengths between 350 and 480 metres, is $0{\cdot}9 \times 10^8$ in E.S.U. While this is considered to be a more accurate determination than has hitherto been made at Slough, it is to be noted that the value of 10^8 was arrived at in the previous experiments as an average value for the South of England.*

The accuracy of the experiments is limited by the effects of surrounding objects, such as trees, on the directions of the forces in the arriving waves, and the fact that in the above table A_0 can depart from 0 by 0·9° is considered to confirm the experience mentioned in Paper 15* that the inclination of the electric force to the vertical, as measured with the portable apparatus, varied in different parts of the field at Slough. In the present case this error is of no serious consequence, since it has been found to be constant and quite negligible compared with the variable changes in the direction of the force.

When observations of the directions of the electric force are carried out in the neighbourhood of sunset, it is found that these directions are subject to changes of a very erratic and sometimes quite large magnitude. In the case of the transmissions from Bournemouth these variations have been observed to begin as much as two hours before sunset in the summer months. The nature of the variations is probably best represented by means of the graphs in Fig. 1, which show the results of a set of observations taken over a period of 2 hrs. 40 mins., the individual observations having been made at half-minute intervals during this time. It is seen that both the angles A and B vary in a rapid and very erratic manner, the period of the approximately cyclical variations being from one to four minutes. During the observations it was noticed that the nature of the signal minimum was continually changing in the case of the angle A as well as B. At one instant the minimum will be quite sharp and well defined, while at a later instant it will become quite blurred, indicating a large rotating component in the particular force being measured. It appears that there is no obvious relation between the sharpness of the minimum and the departure of the angle from its

* SMITH-ROSE and BARFIELD, *loc. cit.*

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normal day value. At times these variations were taking place so rapidly that it was not possible to follow them accurately owing to the large inertia of the rotating Hertzian rod.

It will be noticed that in the case of both A and B the direction of the tilts of the resultant forces are constantly changing sign, and that they are often of reverse sign to the normal direction. The fact that the values of A and B are at times greater than the theoretical value of B_0 is considered to be strong evidence indicating that

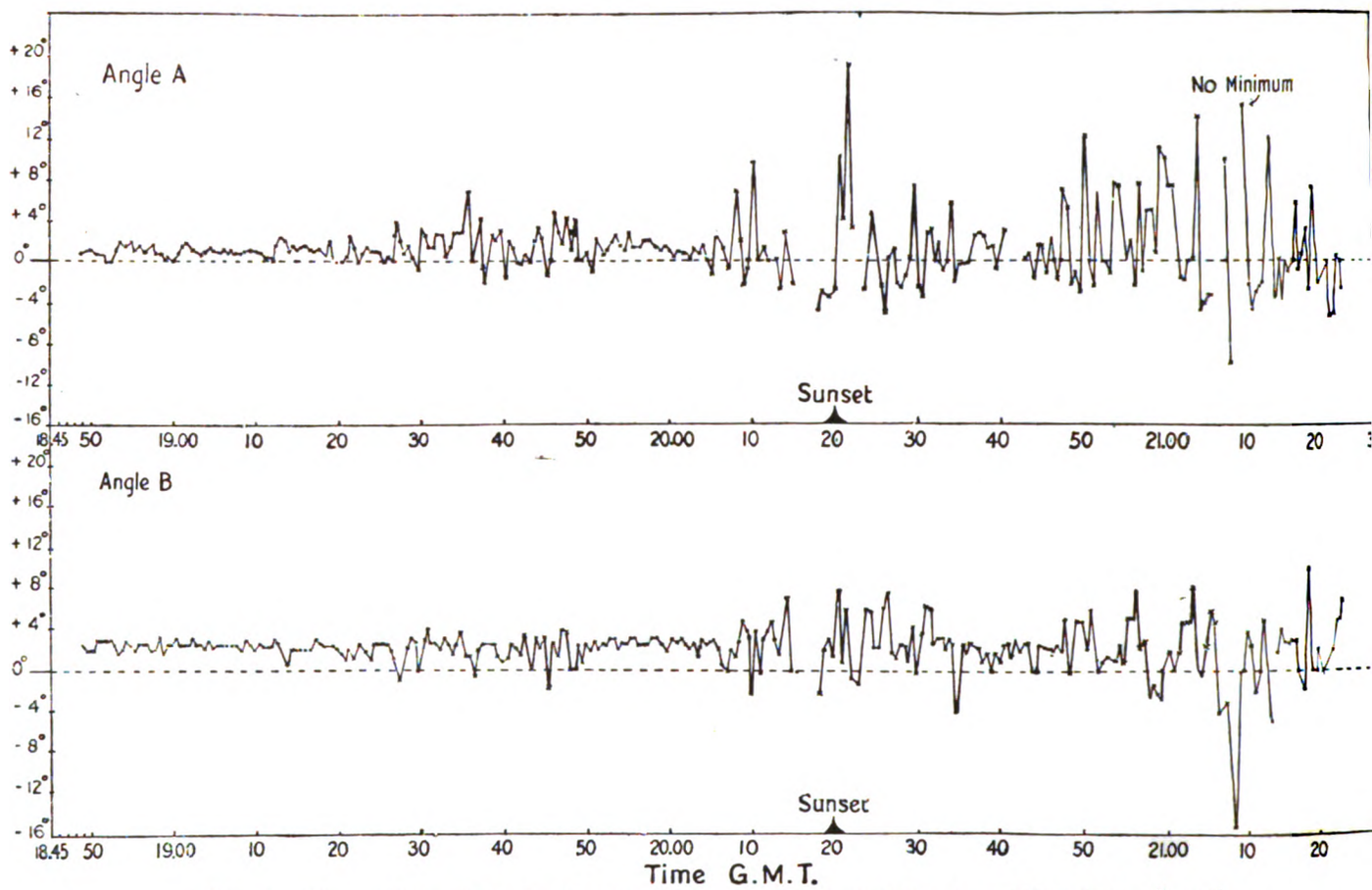

FIG. 1.—Observations of Angles A and B on Bournemouth, 18.6.25. $\lambda = 386$ metres. Sunset, 2020, G.M.T.

during the time of these variations some of the waves received are arriving in a downward direction. When the angle A departs from its normal direction by very large amounts, it would appear that the downcoming wave is largely horizontally polarised, *i.e.*, with the electric force in a horizontal plane; and when B has a large value, the downcoming waves are largely polarised with the electric force in the vertical plane of propagation. Although the observed effects are subject to complications due to variations of phase and amplitude of the arriving waves, the results appear to demonstrate the presence of downcoming waves at night, these waves being polarised in a plane which is almost continually changing or rotating.

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(*b*) *Magnetic Force.*—The frame coil receiver can be used in two ways. First, with the plane of the coil locked in a vertical position, it can be used as an ordinary direction-finder for the determination of the apparent bearing of a transmitting station; the departure of the bearing so obtained from the true great-circle bearing of the transmitter is designated by the angle C. In the stable conditions prevailing during most of the daylight hours, the observed bearing remains constant with an extreme variation of the order of $\pm$ 1°. The error in the mean bearing so obtained (C_0) is the permanent error of the direction-finder under its conditions of use, and is attributable to the local conditions of the site. The values of C_0 for three transmitting stations within the wave-length band employed are given below in Table II.

TABLE II.—Showing Error in Mean Day Bearings on Direction-Finder.

Transmitting station.	Wave-length (metres).	True bearing (degrees).	Error in day bearing C_0 (degrees).
London	365	84·0	+ 1·2
Bournemouth	386	227·4	+ 4·0
Birmingham	479	319·1	+ 2·0

The second manner in which this frame coil receiver may be used is that of rotation about a horizontal axis set either parallel or perpendicular to the direction of propagation of the waves. Under the stable daylight conditions mentioned above, it is found that this rotation produces a zero of signal intensity when the plane of the frame coil is horizontal. This indicates that the magnetic force of the arriving waves is in the horizontal plane, and repeated observations made under daylight conditions have shown that the maximum departure from the horizontal is less than 1°.

When, however, the observations are carried out in the proximity of sunset and during the hours of darkness, the direction of the magnetic force is found to vary in a most erratic manner. Considering first the variations in bearing, or the angle C, these are illustrated in Fig. 2, which shows a typical series of readings taken at half-minute intervals for 2 hrs. 40 mins. during a sunset period. The time at which these variations begin approximately coincides with the beginning of the variations of angles A and B. The observations illustrated were taken at Slough on the Bournemouth transmitting station, and it is considered from previous experience that some coincidence of conditions is responsible for the excessive effects observed. It will be noticed that the apparent bearing varied at times over 180°, and on a few occasions it was noticed that the rotation was continued over a range of 360°. Further observations on other stations and wave-lengths will provide further information for discussion of this point. As has been consistently observed in direction-finding work on wave-lengths in the neighbourhood of 450 metres, these variations are accompanied by a broadening of

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the signal minimum, which indicates that the horizontal component of the magnetic force has become of elliptical or circularly polarised form.

Accompanying the above variations, it is found that measurable tilts of the magnetic force to the horizontal are obtained on these shorter wave-lengths. With the horizontal axis of rotation in the direction of propagation, the angle D, which is the inclination of the coil to the horizontal for the minimum signal position, varies about 4° on either side. When the horizontal axis is set at right angles to the direction of propagation,

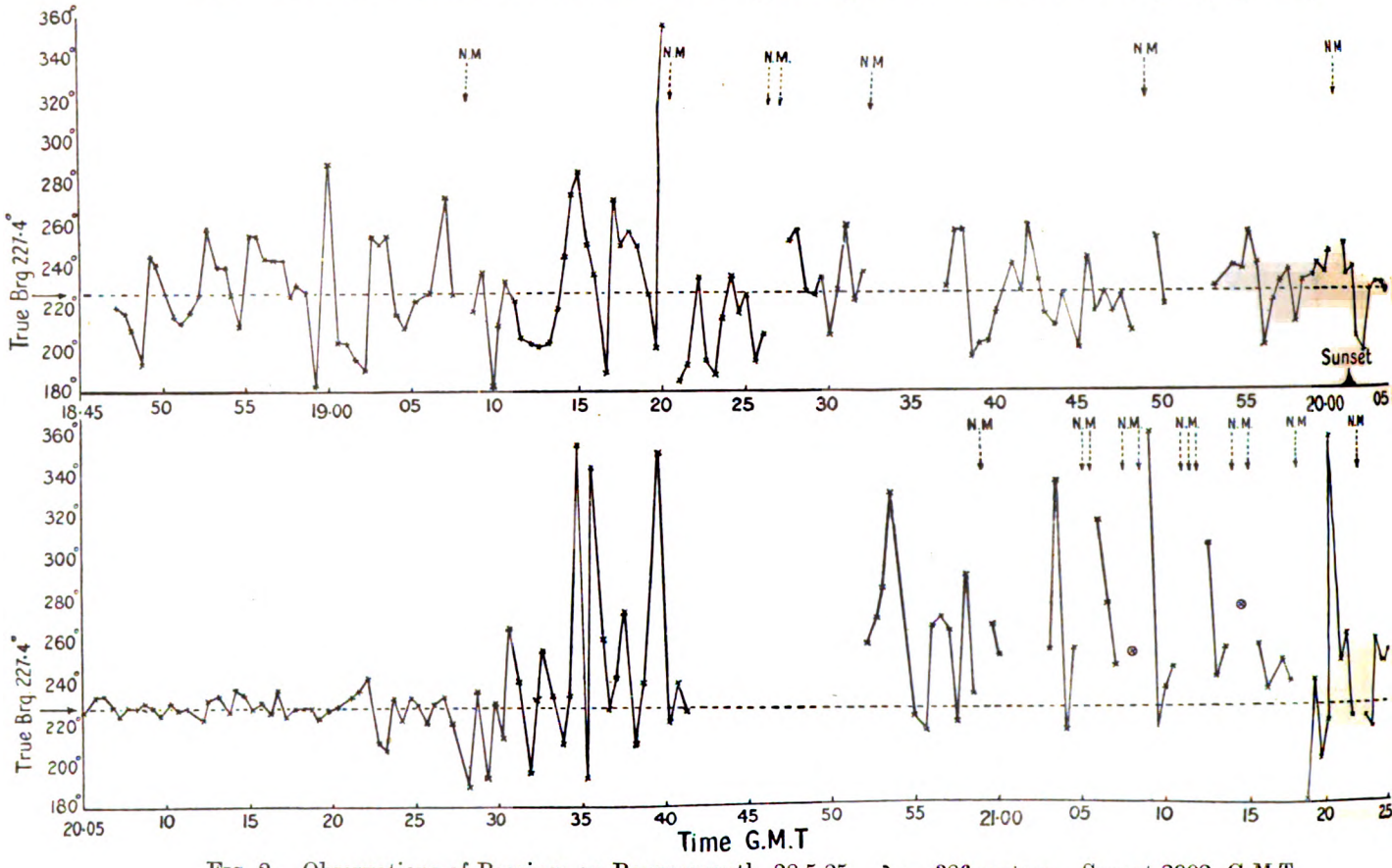

FIG. 2.—Observations of Bearings on Bournemouth, 28.5.25. $\lambda = 386$ metres. Sunset 2002, G.M.T. (N.M. = No minimum.)

the corresponding angle of inclination, F, is usually in the positive direction (see notation later) but varies over the range + 5° to — 6°. These last results are illustrated by the typical curves given in Fig. 3.

In all cases it will be seen that the directions under observation are varying very rapidly, and it is at times difficult to make an accurate measurement. For example, the angle C, error in apparent bearing, has been observed to vary at the rate of 160° in 30 seconds.

(*c*) *Preliminary Conclusions from these Results.*—From the above typical results, which have been confirmed on many occasions, it may be stated that the extension of

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the experiments to shorter wave-lengths in accordance with the conclusions of Paper 15 has been quite successful. It has been shown to be possible to obtain departures of the directions of both the electric and magnetic forces from their normal daylight values, which are measurable by means of the apparatus now employed. This is considered to establish directly the fact that at these times a portion of the energy received from the transmitting station has arrived in a downward direction and thus from the upper atmosphere. From the fact that the variations in the directions of both forces are obtained both in the plane of propagation and at right angles thereto, it may be deduced that the downcoming waves may be polarised either vertically or horizontally. Further, from the values of the angles actually measured, and also from the fact that on the direction-finding coil there is frequently no detectable minimum of signal strength, it may be concluded that the intensity of the wave is of the same order as that of the direct, horizontally propagated, wave.

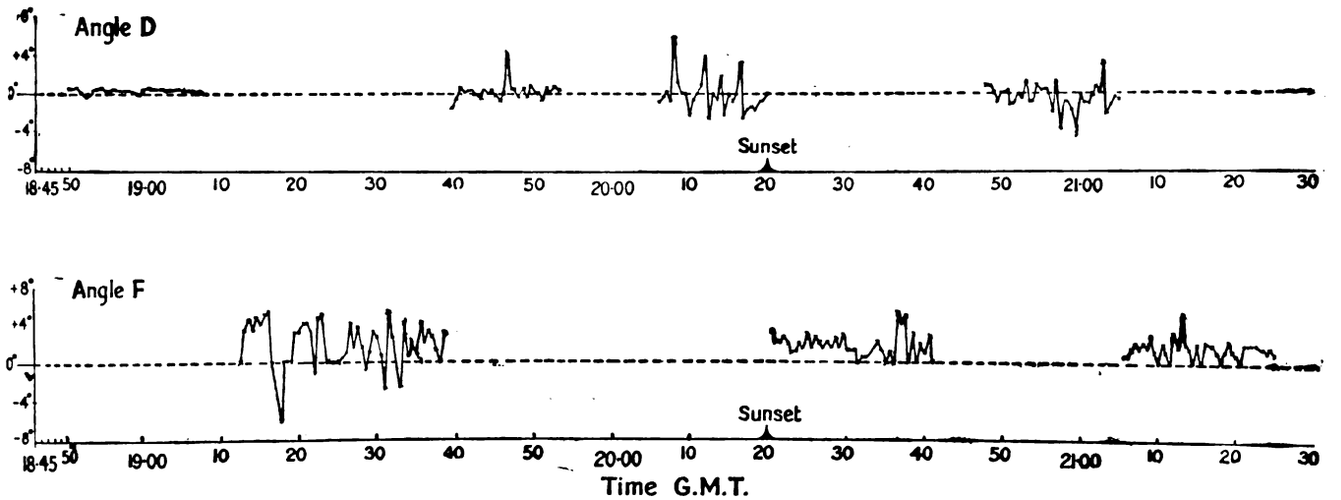

FIG. 3.—Observations of Angles D and F on Bournemouth, 18.6.25. $\lambda = 386$ metres. Sunset, 2020, G.M.T.

(4).—*Development of Apparatus for Additional Measurements on Received Waves.*

The experiments described in the previous section of the paper thus provide a satisfactory *qualitative* answer to the main question underlying the investigation, that is to say, they give conclusive proof (1) of the existence of downcoming waves in certain cases of the same order of intensity as that of the direct wave and arriving at a steep angle of inclination; (2) of the existence in these waves of an important horizontally polarised component. They do not by themselves, however, provide sufficient data for the calculation of the angle of incidence of the downcoming waves, owing to the presence of the direct daytime wave, neither do they provide any means of calculating the relative intensity of the downcoming waves.

It appears, however, as shown in the theoretical discussion in Section 5, that if, in addition, a measurement of the intensity of the resultant vertical electric force under

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normal and abnormal conditions could be made, enough information for a calculation of the angle of incidence would be provided.

As a preliminary to a further advance it was, therefore, decided to develop a suitable method of measuring this component (Z). This was successfully achieved by employing a receiver connected to a tuned vertical aerial and horizontal earth screen, working in conjunction with a three-stage high-frequency amplifier and a detector valve with a sensitive moving coil mirror galvanometer in its anode circuit, the steady anode current through the galvanometer being balanced out by a potentiometer arrangement. The amplifying circuit and galvanometer were calibrated by means of a local valve oscillator adjusted to give a constant output current. As the absolute value of the vertical electric force was not required, there was no need to know the absolute value of the current. It was found with this apparatus that the relative value of Z could be determined to within 10 per cent., while moment-to-moment changes could be faithfully recorded provided the rate of change was not excessive.

The Hertzian rod and signal-strength apparatus together provide sufficient information for the calculation of the angle of incidence of vertically polarised downcoming waves. In order to measure this quantity in the case of horizontally polarised waves, the tilting-coil apparatus, as described in Paper 15, was adapted for working on the shorter wave-lengths now being investigated.

When using this apparatus for the determination of the angle F, the coil is in such a position that the normal direct wave is eliminated. Thus the resulting tilt of the magnetic force to the horizontal is entirely the result of the downcoming waves, and its angle is in fact exactly equal to the *optical angle of refraction* of those waves at the earth's surface (see Section 5). Hence, from the ordinary optical laws of reflection and refraction, the angle of incidence can at once be calculated provided the electrical constants of the earth in the neighbourhood of the receiver be known.

Thus two methods of measuring the angle of incidence of the downcoming waves were available and working satisfactorily. They both, however, suffer from the same two drawbacks; first, that they involve the measurement of residual effects which for longer waves are negligibly small, and which are only just measurable with the wave-lengths now employed; and, second, that they depend on a knowledge of the electrical constants of the ground in the neighbourhood of the receiver, which are by no means accurately known. For these reasons neither of these methods is very accurate.

The need of an alternative method was therefore strongly felt, particularly of a method which should be independent of the value of the ground constants and should not involve the measurement of quantities rendered residual on account of the reflecting power of the earth's surface. It had been realised for some time that such a method existed, but, owing to experimental difficulties among other reasons, it had been temporarily laid aside. It was now thought desirable to develop it, and after a few months' work the method was successfully put into use. The resultant electromagnetic fields produced at the surface of a perfect conductor by a wave travelling

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along that surface and by a wave inclined to it are completely indistinguishable from one another in respect to their directions, but differ in this important respect, that whereas in the former case the electric and magnetic force are equal, in the latter case the magnetic force is greater than the electric force, the ratio of the two intensities being dependent on the angle of incidence of the wave.* As shown in Section 5, if Z be the resultant vertical electric force of the wave of angle of incidence θ, and β its resultant horizontal magnetic force at right angles to its plane of propagation, the relation

$$\frac{\beta}{Z} = \frac{1}{\sin \theta}$$

holds good *whatever the nature of the surface on which the wave is incident and for all wave-lengths.*

The practical method of measuring this ratio and thus obtaining θ is shown in Fig. 4. It consists in comparing the instantaneous value of the signal strength induced respectively in a closed loop and a vertical aerial. The loop and aerial

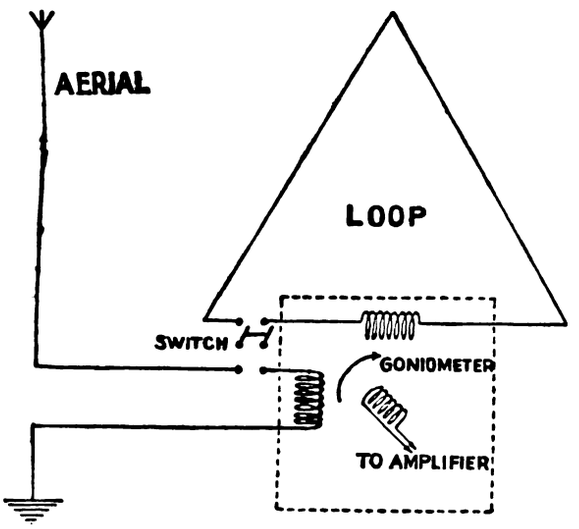

FIG. 4.—Method of Measuring the Ratio Magnetic Field/Electric Field.

are alternately switched on to the respective field coils of a radio-goniometer by means of a switch operated by a single movement. The search coil is meanwhile rotated until a balance is obtained. The tangent of the angle which the search coil now makes with its original position is then proportional to the ratio of the current in the aerial to that in the loop, and this ratio is proportional to β/Z, provided that

* Since the above was written, a paper has been published by APPLETON and BARNETT, 'Proc. Roy. Soc.,' A, Vol. 109, p. 621, 1925, describing experiments in which this principle was used to detect the existence of waves arriving at the receiver in a direction inclined to the horizontal. The experiments showed that the fading of wireless signals was greater on a loop than on an aerial, and from the ratio of the respective variations the angle of incidence of the downcoming waves was calculated.

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the loop is aligned in the direction in which the wave is travelling. The datum value of this ratio is obtained by calibrating the system on a wave which can be safely assumed to be travelling horizontally and for which therefore $Z = \beta$. It must be realised that though this method is superior to the Hertzian rod and tilting coil methods in that it is unaffected by the nature of the earth surface and of the lengths of the waves under investigation, yet in common with the Hertzian rod apparatus it cannot be used to calculate θ without a knowledge of the ratio of the intensity of the direct wave to that of the downcoming wave, and it is therefore necessary, as in the former case, to make simultaneous observations of the relative intensity of the vertical force under normal and abnormal conditions.

In employing the loop-aerial apparatus for the determination of the angle of the downcoming waves, the loop as already mentioned is aligned in the direction of the transmitter. By aligning it at right angles to this direction another ratio can be determined, *i.e.*, the ratio of the vertical electric force (Z) to the magnetic component (α) parallel to the wave direction. This latter component is, of course, zero for horizontally travelling waves, and, in fact, for all vertically polarised waves not laterally deviated. But it becomes finite when there is present a downcoming wave with a horizontally polarised component and in this case the ratio α/Z, though not giving the angle of incidence directly, can give valuable information as to the intensity of the horizontally polarised portion of the waves (see Section 5). In practice, therefore, a second loop was erected at right angles to the first, and by means of a suitable switch the two ratios β/Z, and α/Z were measured alternatively throughout the experiments.

Part II.—An Investigation of the Intensity and Angle of Incidence of Downcoming Waves.

(5).—*Theoretical Discussion of the Effects of Downcoming Waves.*

(*a*) *Notation.*—In order to show the relation between the quantities determinable by the various experimental apparatus above described and the quantities we wish to calculate (*i.e.*, angle of incidence and intensity of downcoming waves), it now becomes desirable to give a brief mathematical discussion of the whole problem. In this discussion the following notation will be used :—

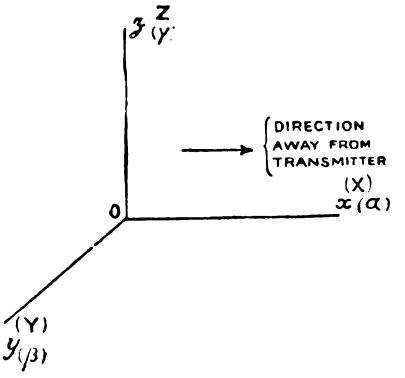

Fig. 5.

It is supposed that the electromagnetic field is under investigation at the origin (O, Fig. 5) of a system of co-ordinates chosen so that :

Ox is along the great-circle path from transmitter to receiver.

Oy is the horizontal direction perpendicular to Ox.

Oz is vertical.

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Let X, Y, Z, be the components of the total electric force at O,
α, β, γ, be the components of the total magnetic force at O,

and let
X′, Y′, Z′, etc., refer to the field components in the ground,
K = dielectric constant of ground,
σ = conductivity in electrostatic units,
n = frequency, and

$$K' = K - \frac{2i\sigma}{n}.$$

This quantity K′ is employed in the analysis whenever the field in the earth itself is under consideration. It then plays exactly the same part with regard to the conducting medium as the dielectric constant K would play in the case of a non-conductive medium.

Thus $\sqrt{K'}$ is the optical refractive index of the earth's surface.

By means of the above notation we can now define more precisely the quantities measured experimentally by the various apparatus employed,* viz. :—

Hertzian rod apparatus determines $X/Z = \tan B$, and $Y/Z = \tan A$.
Tilting coil apparatus determines $\alpha/\beta = \tan C$, $\gamma/\beta = \tan D$, $\gamma/\alpha = \tan F$.
Signal-strength apparatus determines Z (relative values).
Loop and aerial apparatus determines α/Z and β/Z.

In order to find the relation between these quantities and the angle of incidence (θ) of the downcoming wave, several special cases must now be considered.

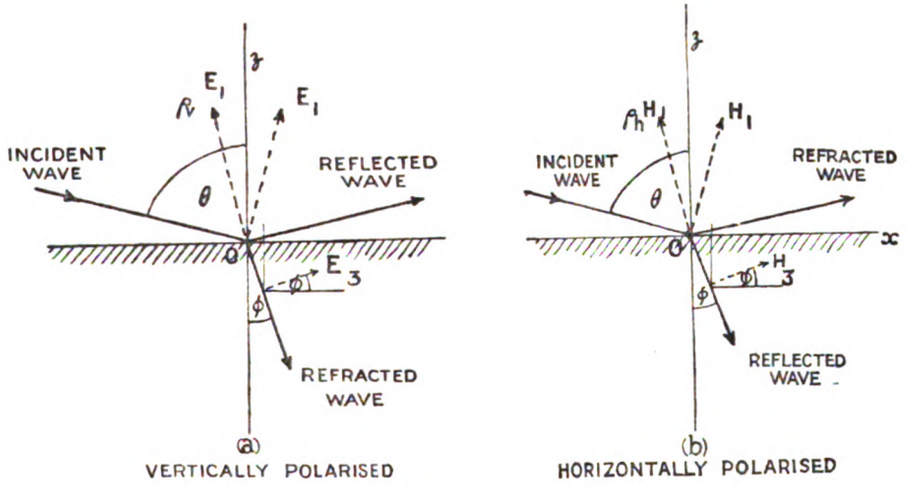

FIG. 6.—Single Downcoming Waves.

(*b*) *Case of a Single Downcoming Wave, Vertically Polarised.*—Let a wave of this description travelling in the plane zOx arrive at angle of incidence θ (Fig. 6). Let E_1

* This assumes the two vectors are in phase and will, therefore, only hold good when the reading is a sharp one.

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be the amplitude of electric field of the incident wave, and let ϕ be the angle of refraction.

Since the wave is vertically polarised, we have

$$Y = \alpha = \gamma = 0.$$

At the earth's surface the boundary conditions demand that the tangential components of force in the two media shall be equal and that the normal components of induction shall also be equal, *i.e.* (assuming for air $K = \mu = 1$; for earth $\mu' = 1$), that

$$\left.\begin{aligned} X &= X' \\ \beta &= \beta' \\ Z &= K' Z' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Therefore

$$\frac{X}{Z} = \frac{X'}{K' Z'} = \frac{1}{K'} \cot \phi. \quad \ldots\ldots\ldots\ldots \quad (2)$$

Now from optical laws and also from standard electromagnetic theory, we have $\sin \theta / \sin \phi = \sqrt{K'}$, so that eliminating θ we get

$$X/Z = 1/K' \frac{\sqrt{1 - \frac{\sin^2 \theta}{K'}}}{\frac{\sin \theta}{\sqrt{K'}}}. \quad \ldots\ldots\ldots\ldots \quad (3)$$

Now under the particular conditions of these experiments $K' \gg 1$, so that the right-hand numerator is approximately unity.

Hence:

$$\frac{X}{Z} = \frac{1}{\sqrt{K'} \sin \theta}. \quad \ldots\ldots\ldots\ldots \quad (4)$$

Now in the special circumstances of a normal "horizontally" travelling wave (*i.e.* the direct wave) there will be no reflected wave.

So that we must have

$$\frac{X_0}{Z_0} = \cot \theta \quad \ldots\ldots\ldots\ldots \quad (5)$$

Eliminating θ in (3) by means of (5), we find that the expression (3) reduces to the exact relation,

$$\frac{X_0}{Z_0} = \frac{1}{\sqrt{K'}}, \quad \ldots\ldots\ldots\ldots \quad (6)$$

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which is identical with ZENNECK's expression for the forward tilt of a wave, and is, in fact, the reciprocal of the refractive index of the medium (see Fig. 7).

We also have from Fig. 6,

$$\left.\begin{aligned} X &= E_1 (1 - \rho_v) \cos \theta \\ Z &= E_1 (1 + \rho_v) \sin \theta \\ \beta &= H_1 (1 + \rho_v) \end{aligned}\right\}, \quad . \quad . \quad (7)$$

where ρ_v = the reflection ratio of the earth at the angle of incidence θ and the frequency of the waves employed.

From (7) we get at once,

$$\frac{\beta}{Z} = \frac{1}{\sin \theta} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (8)$$

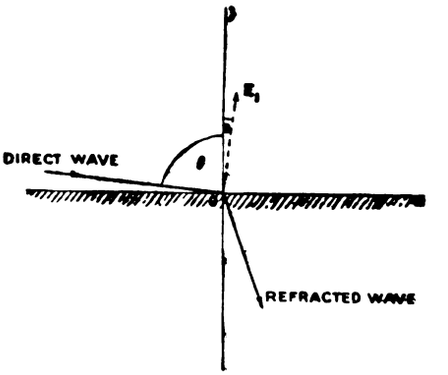

FIG. 7.—Special Case; $\rho_v = 0$ (Direct Wave).

(β/Z, it will be remembered, is the ratio measured by the loop-aerial apparatus described in Section 4.)

(*c*) *Case of a Single Downcoming Wave, Horizontally Polarised.*—In this case $Z = X = \beta = 0$, by hypothesis.

The boundary conditions, as before, require continuity of tangential force and normal induction on crossing the boundary surface, *i.e.*,

$$\left.\begin{aligned} Y &= Y' \\ \alpha &= \alpha' \\ \gamma &= \gamma' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

So that we see at once (since the permeability of both media is unity)

$$\gamma/\alpha = \gamma'/\alpha' = \tan \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But $\gamma/\alpha = \tan F$ (F being the angle measured by the tilting coil as described in Section 4).

Hence

$$F = \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

That is, we are, by this method, making a direct measurement of the angle of refraction of the waves.

The relation between F and θ is given by

$$\tan F = \gamma/\alpha = \tan \phi$$

$$= \frac{\sin \theta}{\sqrt{K'}} \times \frac{1}{\sqrt{1 - \frac{\sin^2 \theta}{K'}}},$$

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therefore

$$\tan F = \frac{\sin \theta}{\sqrt{K'}}, \qquad \ldots\ldots\ldots\ldots \quad (12)$$

under the condition, satisfied throughout these experiments, that

$$\sqrt{K'} \gg 1.$$

(*d*) *General Case of Single Wave.*—If we combine the two preceding cases, we get the general case of a single arriving wave polarised in any plane, or circularly or elliptically polarised. For such a wave it will be seen at once that equations (1) to (12) are still valid.

(*e*) *Case of Two Arriving Waves, Direct and Downcoming.*—In practice, when downcoming waves are being received, there will sometimes be at least two waves arriving simultaneously—the first the direct horizontally propagated wave and the second the downcoming wave (see Fig. 8). Of course there may be two or more downcoming waves, but we will at first consider only one.

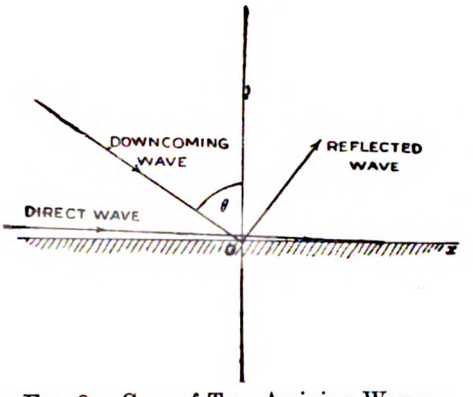

FIG. 8.—Case of Two Arriving Waves.

Let θ be angle of incidence of the downcoming wave. (That of the direct wave = 90° approx.)

Let E_0, X_0, Y_0, Z_0, etc., refer to the electromagnetic field of the direct wave.

X_1, Y_1, Z_1, etc., refer to the resultant field produced by the downcoming wave together with reflected wave.

Then from equations (4), (6), (8), (12),

$$\left.\begin{array}{ll} \dfrac{X_0}{Z_0} = \dfrac{1}{\sqrt{K'}}; & \dfrac{X_1}{Z_1} = \dfrac{1}{\sqrt{K'}\sin\theta} \\ \dfrac{Z_0}{\beta_0} = 1 \text{ (approx.)}; & \dfrac{Z_1}{\beta_1} = \sin\theta \\ \dfrac{\gamma_0}{\alpha_0} = 0/0; & \dfrac{\gamma_1}{\alpha_1} = \dfrac{\sin\theta}{\sqrt{K'}} \end{array}\right\} \ldots\ldots\ldots \quad (13)$$

We also have the vectorial relation:

$$\left.\begin{array}{l} X = X_0 + X_1 \\ Z = Z_0 + Z_1 \\ \alpha = \alpha_0 + \alpha_1 \\ \beta = \beta_0 + \beta_1 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \quad (14)$$

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So that from (13) and (14) we finally obtain the following three independent expressions for sin θ :—

$$\text{(Hertzian rod quantities) } \sin\theta = \frac{1}{\sqrt{K'}}\left(\frac{Z - Z_0}{X - X_0}\right), \quad \ldots\ldots\ldots \quad (15)$$

$$\text{(Tilting coil quantities) } \sin\theta = \sqrt{K'}\left(\frac{\gamma - \gamma_0}{\alpha - \alpha_0}\right) = \sqrt{K'}\,\gamma/\alpha, \quad \ldots \quad (16)$$

$$\text{(Loop and aerial quantities) } \sin\theta = \left(\frac{Z - Z_0}{\beta - \beta_0}\right), \quad \ldots\ldots\ldots\ldots \quad (17)$$

where the magnitudes of all the vectors on the right-hand side of these equations are determinable by experiment, in some cases directly and in others indirectly.

Of these three methods it will be noticed the third is independent of K′, the second is independent of the nature of the direct wave, whilst the first has neither of these advantages. To obtain a numerical value for θ from (15) and (17), it is necessary to know the phase relation between the downcoming and direct waves. As it is quite impossible at present to measure this angle directly, it is necessary to select from the experimental results only those observations for which a definite assumption of its value can be made. The discussion of how far this can be done and how it affects the accuracy of determination of θ is conveniently left until the section dealing with the actual results obtained (p. 353).

It is further to be noted that of the above three methods of measuring the angle of incidence, the second, making use of equation (16), applies only to the horizontally polarised portion of the downcoming waves, while the other two apply to the vertically polarised waves.

(*f*) *Relative Intensities of Direct and Downcoming Waves.*—It is also possible from these equations to find simple expressions for the relative intensity of the downcoming wave in terms of the intensity of the direct wave. Thus, for the vertically polarised component :—

$$E_1 = \frac{Z_1}{(1 + \rho_v)\sin\theta},$$

from (7), and $E_0 = Z_0$ approximately, so that

$$\frac{E_1}{E_0} = \left|\frac{Z - Z_0}{Z_0\,(1 + \rho_v)\sin\theta}\right|, \quad \ldots\ldots\ldots\ldots \quad (18)$$

and for the horizontally polarised component, by reasoning similar to that from which (7) was obtained,

$$\begin{aligned}\alpha_1 &= H_1\,(1 + \rho_h)\cos\theta \\ &= E_1\,(1 + \rho_h)\cos\theta,\end{aligned}$$

where ρ_h is the reflection ratio of the earth for horizontally polarised waves arriving at angle of incidence θ.

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Also

$$E_0 = Z_0 = \beta_0 \text{ (approx.)}.$$

Thus

$$\frac{E_1}{E_0} = \frac{\alpha_1}{Z_0}(1 + \rho_h) \cos \theta, \quad \ldots\ldots\ldots\ldots \quad (19)$$

$$\alpha_1 = \alpha - \alpha_0 = \alpha.$$

$$\frac{E_1}{E_0} = \alpha/Z \times Z/Z_0 \times \frac{1}{(1 + \rho_h) \cos \theta}. \quad \ldots\ldots \quad (20)$$

All the quantities on the right-hand side of (19) and (20) can either be determined experimentally or calculated from known data. The reflection ratios ρ_v and ρ_h are determined from the simple FRESNEL's optical reflection formulæ.

$$\rho_v = \frac{\tan (\theta - \phi)}{\tan (\theta + \phi)}, \quad \ldots\ldots\ldots\ldots \quad (21)$$

$$\rho_h = \frac{\sin (\theta - \phi)}{\sin (\theta + \phi)}, \quad \ldots\ldots\ldots\ldots \quad (22)$$

where ϕ is the angle of refraction given by

$$\frac{\sin \theta}{\sin \phi} = \sqrt{K'}.$$

(*g*) *Case of More than one Downcoming Wave.*—While in the above analysis we have assumed that there is only one downcoming wave, it is clearly necessary to envisage the possibility of more than one. In such a case the field components E_1, X_1, Y_1, etc., of the above equations must be taken as representing the sum of the individual components due to all the downcoming waves present. If all these waves arrive with the same angle of incidence there is obviously no difference between this case and the former case, and there is no need to distinguish between them, nor, in fact, possibility of doing so. If, however, they are arriving at different angles of incidence, θ_1, θ_2, θ_3, etc., the angle θ determined from equations (15), (16), (17), will represent only a mean or resultant angle of incidence. This mean will naturally be biassed towards the wave of greatest intensity. It will also be affected by the relative phases of the component waves, and in certain critical cases (*i.e.*, when the vector sum of the intensities approaches zero) it can conceivably give an entirely misleading or impossible value for θ—a value, for instance, not even lying between θ_1, θ_2, θ_3, etc. The discussion of the possibility of eliminating such misleading cases must again be left until the experimental results are under consideration (see Section 8).

(*h*) *Conclusions from the above Analysis.*—From this analysis the conclusions were

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therefore drawn that the angle of incidence of the resultant downcoming wave could be obtained with the apparatus available in three independent ways, viz. :—

1. By the Hertzian rod apparatus, provided the ground constants (refractive index) were known and that a definite value could be assumed for the phase difference between the direct and downcoming waves.
2. By the tilting coil apparatus, provided the ground constants were known, but *without any knowledge of the nature or phase of the direct wave.*
3. By the loop-aerial method, provided the phase assumption can be made, but *independently of the value of the ground constants.*

Since no two of these three methods are dependent on the same proviso, they clearly form valuable checks on each other.

The analysis further shows that the relative intensity of the resultant downcoming wave can be calculated when once θ has been determined.

Of the assumptions underlying the analysis only one was at all open to doubt. The nature of this assumption and the means by which it was checked are dealt with in the next section (6).

(6).—*Investigation of Lateral Deviation of Arriving Waves.*

One of the assumptions on which the foregoing analysis is based has had hitherto little or no experimental evidence to support it. This assumption is that all waves arriving at the receiver are propagated in the plane of $z\,O\,x$ = the vertical great-circle plane between transmitter and receiver. If there were any serious lateral deviation from this plane, the simple relationship between the angle of incidence and the quantities measured (equations 15, 16 and 17) would no longer hold, and further analysis shows that it would be extremely difficult, if not impossible, to determine θ under such conditions.

An experimental verification of this assumption is therefore highly desirable. In the case of a wave entirely polarised in a vertical plane, an ordinary rotating-coil direction-finder suffices to determine the angle of lateral deviation (ψ), but this method breaks down if the wave has a component polarised in a horizontal plane.

Although devised chiefly with a view to reducing interference or the variable errors of direction-finders, several arrangements have been described which provide methods of measuring ψ, but though sound in theory they are surrounded by many experimental difficulties. It was to the overcoming of these difficulties that attention was directed in the present investigation, and after a considerable amount of research in this direction one of ADCOCK's suggested methods* was made to work with sufficient accuracy for the purpose in hand.

* F. ADCOCK, British Patent 130,490 (1919).

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The arrangement comprised four vertical-wire Hertzian receivers 40 ft. high, erected at the four corners of a square of 20 ft. side, with leads from their centres brought to the field coils of a goniometer, which, with the rest of the receiving apparatus and operator, were in a hut above the ground so as to be as near as possible to the electrical centre of the system. The two aerials at the extremity of each diagonal formed a pair, connected to one of the field coils but in opposite senses, so that at any instant the current through the field coil due to one of the aerials in the pair would be in the opposite direction to that produced by the other member of the pair.

A simple consideration of this arrangement shows that the ratio of the currents in the field coil of the goniometer gives the tangent of the angle of lateral deviation, and this ratio, as in the ordinary Bellini-Tosi D.F. system, is given by the tangent of the scale angle of the search coil. If there is more than one wave arriving, the method will give a mean angle of lateral deviation.

A series of experiments were made on those transmitting stations for which θ was measured, and the result substantially confirmed the assumption made in the analysis, that the lateral deviation was negligibly small.

It may be pointed out that if it can be satisfactorily demonstrated that the arriving waves are not subject to any lateral deviation at periods when errors are obtained on a direction-finder, this would constitute another independent proof of the arrival of downcoming waves and of the rotation of the plane of polarisation of such waves.

(7).—*Description of Experiments made on Transmissions from Bournemouth Station.*

We have seen from Section (4) that four sets of apparatus have been devised for measuring separate quantities connected with the resultant electromagnetic field of the arriving waves, and that these together provide three methods of measuring the angle of incidence of the downcoming wave; also that the intensity of the downcoming wave can be determined at any instant when the angle of incidence is known.

After some preliminary experiments carried out during the early months of 1925, it was decided to make all four sets of observations simultaneously. All four sets of apparatus were accordingly set up in the large field at the Radio Research Station at Slough, at distances from each other of about 100 yards. A system of routine observations was started in May, 1925, concentrating on the B.B.C. station at Bournemouth, 77 miles distant, operating on a wave-length of 386 metres. On one day per week simultaneous observations were taken on all four sets of apparatus for a period of about an hour and a half some hours before sunset, and for a period of about three hours extending through sunset. The first of these periods enabled the daytime values of the quantities being determined to be established (these values, as has been seen, being essential to the determination of θ), and also gave a satisfactory idea of the limits of accuracy of the apparatus. The second of these periods covered the transition

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period from the normal daytime conditions to the state of affairs brought about by the arrival of downcoming waves, and thus gave results which, coupled with those obtained in the first period, enabled the angle of incidence of the downcoming waves to be calculated. Simultaneous readings were taken on each instrument every half-minute during the period of working, great care being taken to synchronise the watches of the four observers. But in addition to this an occasional rapid series of observations was made in which readings were taken once every five seconds, in order, in the first place, to obtain a rough idea of the maximum rate at which variations were taking place in the quantities being measured, and, in the second place, to obtain better synchronisation of the observations.

(8).—*Summary of Results obtained on Bournemouth* ($\lambda = 386$ *metres*) *from May to July*, 1925.

After the observations had been made, they were recorded in the form of curves. Each quantity was plotted on a separate sheet, but the time scale on each sheet was the same in order that the variations could be readily compared. During the period under discussion a large number of data were collected, and while it is, of course, not necessary to reproduce all the curves, a selection has been made and is reproduced in this paper in order to illustrate the general nature of the phenomena observed and to give an example of the data from which the angle of incidence of the downcoming waves was deduced.

(*a*) *Curves of Z.*—The daytime value was found to be steady to within 10 per cent., and as it is not certain whether the variations within this limit are due to actual signal-strength variations or to instrumental causes, this must be taken as the limit of accuracy of the apparatus. The observations in this case were made every fifteen seconds.

Fig. 9 shows results obtained at night, the observation being taken at intervals of 30 seconds. Variations begin to be appreciable at least one hour before sunset. The variations are small at first, but increase steadily until they reach a maximum amplitude of about 70 per cent. of the day value. The fading oscillations show a tendency to be asymmetrical about the mean, the positive peaks being greater than the negative.

Fig. 10 shows the results of a rapid test in which readings were taken every five seconds for a period of five minutes; the curve is plotted on the same sheet as similar curves, taken simultaneously, of the value of β/Z. The curve of Z shows that during this particular period the rate at which variations took place was on the average from 10 to 20 per cent. in 5 seconds, while on the other hand on one occasion it was as great as 100 per cent. in 5 seconds.

(*b*) *Curves of Angle B.*—In Fig. 1 were shown curves of the value of the angle B taken simultaneously with those of Z in Fig. 9. The day value of the angle B remains constant to less than a degree, except for isolated little peaks. The night values

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(Fig. 1) show signs of appreciable changes at about the same time before sunset as in the case of Z. The preliminary variations are followed by a distinct " lull " just before sunset, as in the case of Z, after which they start again and increase to their maximum. They are distinctly asymmetrical, the tendency always being for the tilt to increase. Negative values of B are not very frequent, but they certainly occur from time to time. The tilt often exceeded 5°, on a few occasions rose to 8°, and once to 15·5°. On comparing these figures with Z curves obtained simultaneously, no definite correspondence can be traced.

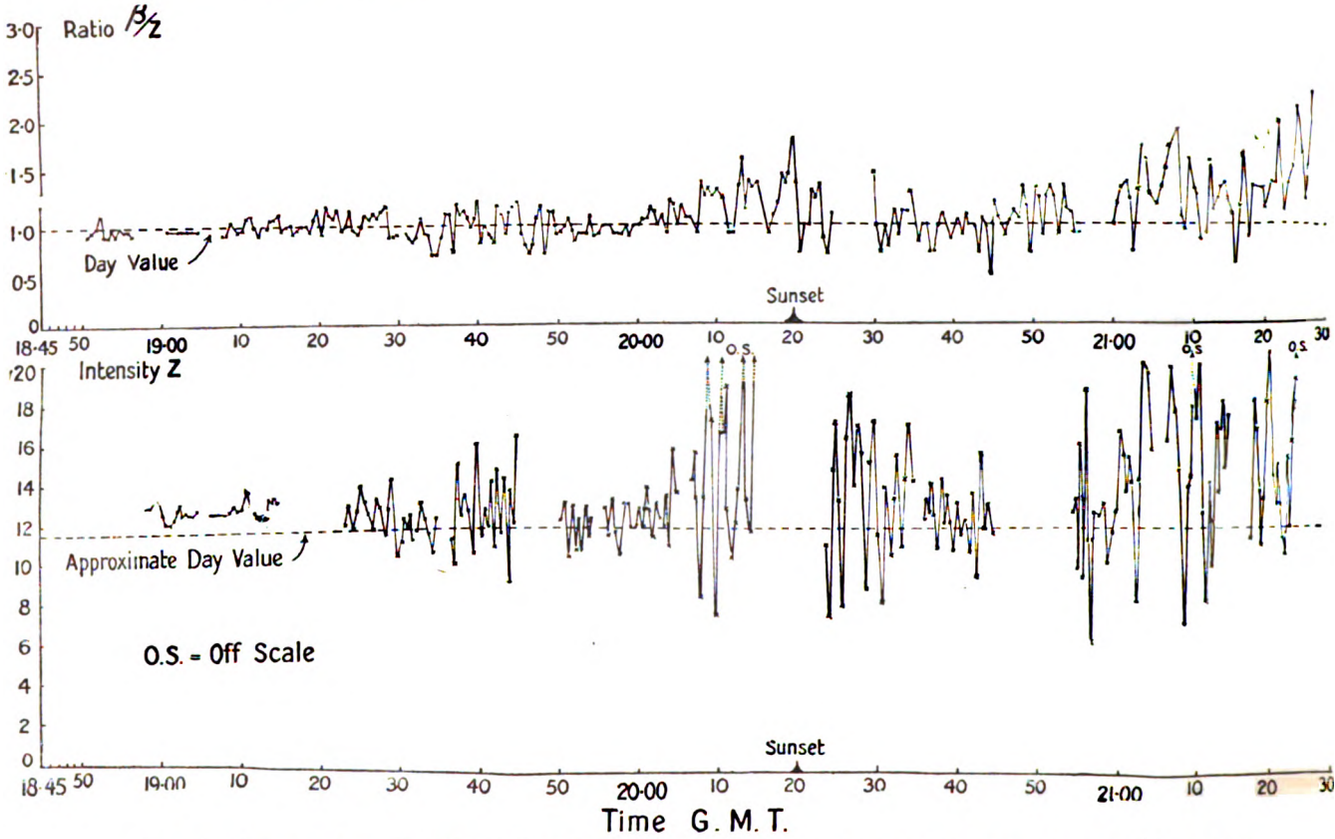

FIG. 9.—Observations of β/Z and Z on Bournemouth, 18.6.25. λ = 386 metres. Sunset 2020, G.M.T.

No rapid tests could be taken of the value of B, for owing to the inertia of the beam carrying the aerial, it was impossible to take readings once every five seconds. It is therefore difficult to gauge the rate at which the variations were taking place, but as far as could be judged they were too quick to be followed continuously.

(*c*) *Curves of Angle F* ($\tan^{-1}\gamma/\alpha$).—Fig. 3 showed an example of the results obtained from a measurement of the angle F. This angle cannot be measured in the day-time, as the signal strength is zero for all inclinations of the coil. At night it is only possible to obtain a reading when the signal strength in the day minimum position of the coil is of the same order as the strength in the maximum position, *i.e.*, when the minimum is very flat or the error in bearing very large. This figure illustrates well the general

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results obtained. The variations are almost all confined to the positive side of the zero, corresponding to a forward tilt of the magnetic field. The variations rarely exceed + 4°. There is a distinct tendency for the angle to settle down at a steady value of about + 2°, even though during this steady state Z, B, and β/Z may be varying greatly. Reference to the α/Z curves show that this steady value is invariably accompanied by large values of α/Z of the same order as β/Z.

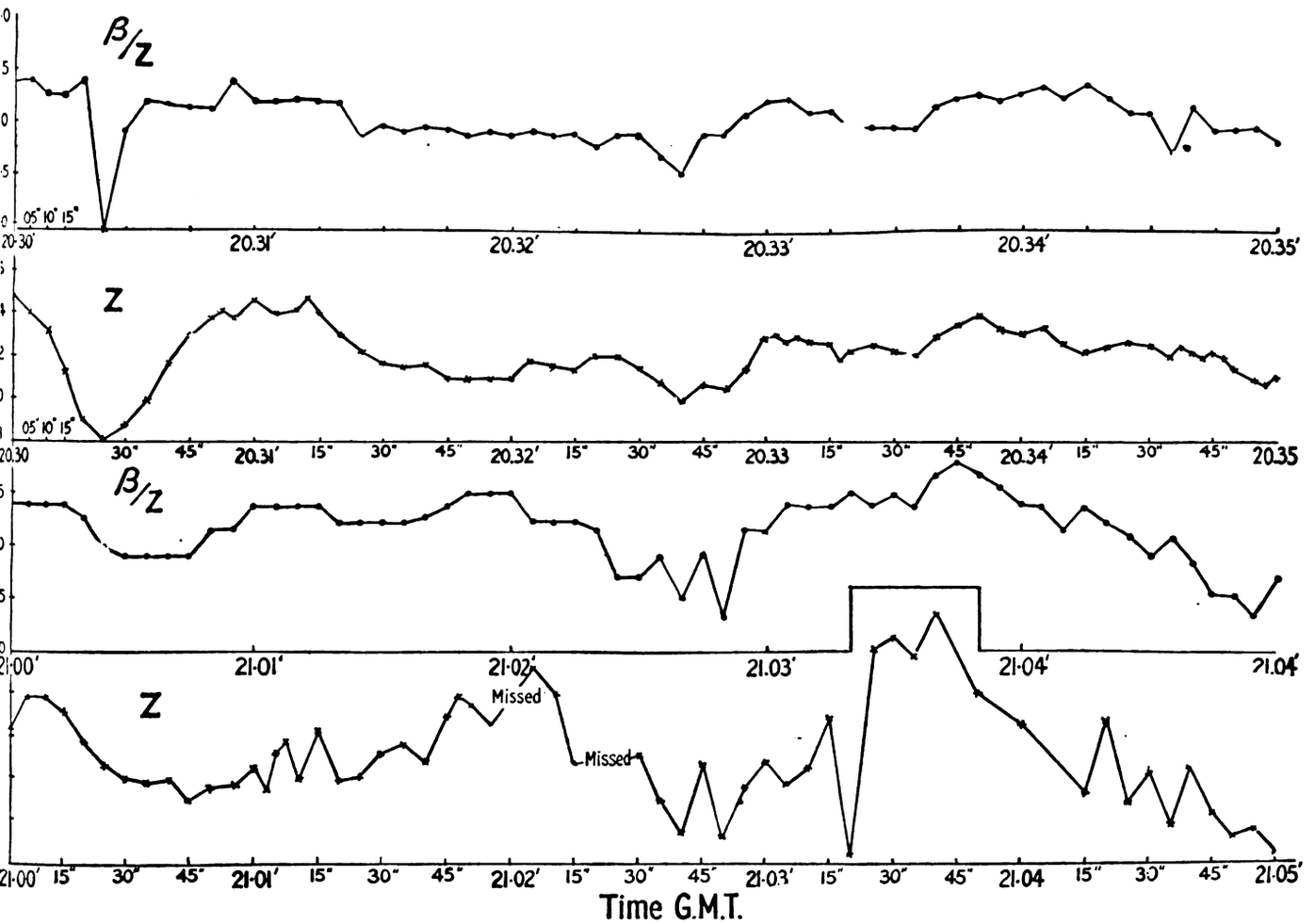

FIG. 10.—Simultaneous Observations of β/Z and Z on Bournemouth, 25.6.25. λ = 386 metres. Sunset, 2021, G.M.T. Special Rapid Tests. Readings every 5 seconds.

The variations of F are unique in that they tend to become small when the others, B, Z, β/Z, α/Z, etc., increase.

It is interesting to point out here that, in seeking for evidence of a downcoming wave, ECKERSLEY* effectively measured the ratio γ/α by comparing the E.M.F.'s induced in a horizontal and in a vertical loop by the incoming waves at such times as the latter gave an error in the apparent bearing of the transmitting station. Experi-

* T. L. ECKERSLEY, 'Radio Review,' Vol. 2, p. 234, 1921.

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mental results obtained at a receiving station near Cairo indicate that, on a wavelength of 2,600 metres, the magnetic field of the arriving waves was tilted at an angle which was not more than 2° from the horizontal at times when the usual night effects were present.

(*d*) *Curves of Ratio* β/Z.—Figs. 9 and 10 show typical curves obtained from a measurement of the ratio β/Z. The day variation of this quantity was found to be within 10 per cent., and this may be taken as the limit of accuracy of the apparatus. Night variations (Fig. 9) start about an hour before sunset, as is the case with the other quantities investigated, and in this instance also the variations steady down just before sunset. The night variations, when small, are roughly symmetrical about the mean daytime value. As, however, they get larger they become distinctly unsymmetrical, the tendency always being for β/Z to be greater than unity (its normal value). Nevertheless, on rare occasions the value dips almost to zero. In Fig. 10 is shown the result of a rapid test carried out simultaneously with observations of Z, the curve of which is given on the same diagram. The two curves show very satisfactory agreement, the two values quite clearly increasing and decreasing together. This indicates that the observations were quick enough to follow the variations of both Z and β/Z continuously.

(9).—*Calculation of the Angle of Incidence of Downcoming Waves.*

(*a*) *Hertzian Rod Method* (*Angle B*).—As has been shown in the theoretical analysis (Section 5), the relation between θ and the quantities measured by the Hertzian rod apparatus is given by

$$\sin\theta = \left|\frac{1}{\sqrt{K'}}\right| \times \left|\frac{Z - Z_0}{X - X_0}\right| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This relationship, however, only holds good at the surface of the earth, whereas the centre of the Hertzian rod which measured B was 20 feet above the ground. This fact necessitates the introduction of an important correction factor C_h, which allows for the phase displacement which must exist between the reflected and incident waves at a point above the ground.

The relation (15) thus becomes

$$\sin\theta = \left|\frac{1}{\sqrt{K'}}\right| \times \left|\frac{Z - Z_0}{X - X_0}\right| \times C_h . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

C_h is a simple function of the height above the ground, the wave-length, and the angle of incidence. For the particular height in question (*i.e.*, 20 feet) the value of C_h varies from about 1 to 2·5 as $\left|\frac{Z - Z_0}{X - X_0}\right|$ decreases from 16·5 to 2·5.

If the angle B is sharply defined, it may be assumed that

$$\tan B = X/Z \text{ approx.},$$

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and also that the two waves are approximately in phase; so that the vector quantities in (15*a*) can be treated as scalar quantities in order to obtain a numerical value for sin θ.

The constant $1/\sqrt{K'} = \sqrt{n/2\sigma}$ is obtained from a knowledge of the conductivity of the ground as actually measured by the apparatus.

Taking $\sigma = 10^{9}$ E.S.U. for Slough, and $n = 0{\cdot}78 \times 10^{6}$ cycles per second for Bournemouth,

$$\frac{1}{\sqrt{K'}} = 0{\cdot}062,$$

so that

$$\sin\theta = 0{\cdot}062 \times \frac{Z - Z_0}{X - X_0} \times C_h.$$

To proceed further we must know Z and Z_0, when X and X_0 can be calculated.

Z_0 is the day value of the signal strength, while Z is the value at the instant when B is measured. X is obtained from the equation $X = Z \tan B$ and X_0 is obtained from equation (6).

Unfortunately the variations of both B and Z were too rapid to enable the value at any instant to be compared. In order to make this possible it would be necessary to take readings at the rate of one per 5 seconds simultaneously on both instruments. This was actually done in the case of Z, but the Hertzian rod operator could not take the reading more rapidly than once per half minute. In these circumstances the best that can be done is to assume that at a given value of B the value of Z did not exceed a certain maximum value (Z maximum), which can easily be fixed by taking the highest value of Z occurring during a few minutes on either side of the instant in question. In this case we get a maximum value for θ, for it can be shown that

$$\sin\theta < 0{\cdot}062 \times \frac{Z - Z_0}{X - X_0} \times C_h,$$

the value X being obtained on the assumption Z = Z maximum.

Under these conditions the value of B which leads to the smallest maximum value of θ will be most interesting; so to make the calculation we shall naturally select the peak value of B. Intermediate values are of no particular interest.

The value of $\theta_{max.}$ obtained by taking a random selection of peak values of B during the investigation are shown in Table III.

(*b*) *Tilting Coil Method* (*Angle F*).—The theoretical analysis shows that in this case the angle of inclination can be obtained from equation (12).

$$\sin\theta = \gamma/\alpha\ \sqrt{K'} = \sqrt{K'} \tan F. \quad \ldots\ldots\ldots\ldots \quad (12)$$

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TABLE III.—Maximum Angle of Incidence calculated by Angle B and Intensity Z.

Date.	Time. G.M.T.	B degrees.	Z max.	θ max. degrees.
28.5.25	1852¼	6	16	38
28.5.25	2115	12	18	19
28.5.25	2052½	7	16	32
28.5.25	2120¾	9	16	22
11.6.25	2111¾	9·4	18	25
18.6.25	2023¾	6	19	44
18.6.25	2026¼	8	19	32
18.6.25	2056¼	8	19	32
25.6.25	2103	7	20	38
25.6.25	2049	8	17	23

It is clear that to get a real positive value for θ the conditions are

$$0 < F < \tan^{-1} 1/\sqrt{K'},$$

$$\text{or } 0 < F < 3{\cdot}5^\circ,$$

(under these conditions of wave-length and conductivity).

A glance at the curves of F (Fig. 3) shows that this condition is fulfilled in the majority of cases, but that there are occasions when F becomes negative, and also others when it becomes greater than 3·5°. The presence of these "impossible" values for F at once shows that we are not dealing with a single wave, the alternatives being either a small lateral deviation of the vertically polarised wave from the plane zOx, for which α would then have a finite value, or the existence of at least two downcoming waves. Which of these two alternatives is the correct one must remain a matter of doubt until more accurate measurements of lateral deviation have been rendered possible. In either case it is quite impossible to obtain a value of θ from these F curves alone, but in both cases, if we choose a moment when α is large—say of the same order as β_0—the chance of obtaining a false value for θ becomes negligible. For since lateral deviation, if it exists, is small, a large value for α means that the preponderating part of this component is produced by the downcoming wave, and hence will alone determine F, and the two or more waves, if they exist, must be combining (*i.e.*, approximately in phase) when α is large. Here again F will give a true value of θ for the resultant downcoming wave.

Thus, to obtain F we must choose those instants when by inspection of the simultaneous curves of α/Z and Z we see that α is of the same order as β_0. If this is done, it is found that at periods of large α the value of F tends to settle down to an average value of about 2° and has no "impossible" values.

From this we at once get

$$\sin\theta = \sqrt{K'} \tan F = 16{\cdot}2 \times 0{\cdot}035.$$

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Hence

$$\theta = 34°.$$

The accuracy of the result is largely dependent on the accuracy with which we know K′.

Although there is a remarkable tendency for the value of F to remain at about 2°, it sometimes has other values when the value of α/β_0 is sufficiently large for a calculation to be made. A selection of such values has been made and the values of θ calculated from them are given in Table IV, assuming $\sigma = 10^8$.

TABLE IV.—Calculation of θ from Angle F. Assume $\sigma = 10^8$.

Date.	Time G.M.T.	α/β_0.	F degrees.	θ degrees.	E_1/E_0.
18.6.25	2107½	2	1·6	28	1·2
18.6.25	2111½	1	(0)*	(0)	0·5
18.6.25	2122½	1	2·0	34	0·6
18.6.25	2124	1	1·2	19	0·5
25.6.25	2045	0·5	2·0	34	0·3
25.6.25	2051½	0·5	(0)*	(0)	0·3
25.6.25	2112½	>2	1·0	16	>1·1
25.6.25	2113	>2	1·0	16	>1·1
25.6.25	2113½	>2	2·0	34	>1·2
25.6.25	2114	>2	2·0	34	>1·2

* The accuracy of this method decreases very rapidly as F decreases to zero, and the values of θ obtained in these two cases are probably due to this limitation.

In concluding this section it is to be observed that the calculation of θ from the measurements of the angle F refers only to the horizontally polarised portions of the downcoming waves. The other two methods, the results of which are discussed in sub-sections (*a*) and (*c*), give the angle of incidence of the vertically polarised portion of the downcoming waves.

(*c*) *Loop-Aerial Method.*—The relation between the angle of incidence and the quantities measured by the loop-aerial and signal-strength apparatus is given by equation (17) of the theoretical analysis (Section 5), viz. :—

$$\sin\theta = \left|\frac{Z - Z_0}{\beta - \beta_0}\right| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The ability to get a numerical value for θ by this relation from the information contained in the β/Z and Z curves (*e.g.*, Fig. 9) depends first on whether the variations were slow enough to enable corresponding values of β/Z and Z to be selected at any given instant, and secondly on whether we can make any definite assumption with regard to the phase relationship of the two waves. As regards the first point, a close

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inspection of the curves shows that only in the case of the rapid tests (see Fig. 10) is it possible to make sure of obtaining for a value of β/Z at a given instant the corresponding value of Z at the same instant. We can, however, utilise the slower tests, in the same way as with the B curves, to obtain maximum values of θ. As regards the second point, a careful consideration of the whole problem leads to the conclusion that it is safe to assume the direct wave and resultant downcoming wave are approximately in phase when both β/Z and Z have attained peak values. This would not be possible but for the fact that a phase difference up to about 60° makes very little difference in the results. With this assumption we can treat the vector quantities in (17) as scalar quantities and thus obtain a numerical value for θ.

The values of θ for various instants during the rapid tests are given in Table V, while the following is an example of how the values were calculated and the possible error estimated :—

Example.—On June 25, 1925, at 2103·40″ (Fig. 10).

$$Z = 19{\cdot}7 \text{ and } \beta/Z = 1{\cdot}68.$$

Therefore

$$\beta = 33{\cdot}1.$$

From day curves of Z on same day

$$Z_0 = 11{\cdot}5. \qquad \beta_0 = 11{\cdot}5.$$

The waves are approximately in phase as both Z and β/Z are at peak values.

$$\sin\theta = \left|\frac{Z - Z_0}{\beta - \beta_0}\right|$$

$$= \frac{8{\cdot}2}{21{\cdot}6}.$$

Hence

$$\theta = 22{\cdot}5°.$$

Possible Errors.—Assuming Z and β/Z can be measured to $\pm$ 10 per cent.—

Max. possible error in Z causes $\pm$ 1·5° change in θ.

Max. possible error in β/Z causes $\pm$ 4·0° change in θ.

The waves are assumed to be in phase. It can be shown that large variations in phase have a small effect on the derived value of θ for given values of Z and β/Z. For example, even if the phase difference was 45°, θ would only be increased by about 1·5°. The maximum possible error is, therefore, about $\pm$ 6°, and the probable accuracy is far higher than this.

The values of θ in Table V are more satisfactory than those of $\theta_{max.}$ in Table III, as they are definite measurements of the angle of incidence of the resultant downcoming waves at each particular moment. The limiting values of θ are, however,

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TABLE V.—Calculation of Angle of Incidence from β/Z and Z (Rapid Tests).

Date.	Time G.M.T.	Z.	β/Z.	θ degrees.	E_1/E_0.
18.6.25	2030·00	17	1·6	21	0·7
18.6.25	2032·50	15·2	1·3	27	0·4
18.6.25	2034·40	16·5	1·4	25	0·6
18.6.25	2101·10	16·5	1·6	21	0·7
18.6.25	2104·05	20	1·9	19	1·2
18.6.25	2104·25	19·6	2·1	17	1·4
25.6.25	2030·00	14·8	1·4	21	0·4
25.6.25	2100·00	14·2	1·4	19	0·4
25.6.25	2103·40	19·7	1·7	23	1·1

TABLE VI.—Calculation of Maximum Angle of Incidence from β/Z Curves.

Date.	Time G.M.T.	$Z_{max.}$	β/Z.	β.	$\theta_{max.}$ degrees.
28.5.25	1912½	15	1·5	22	20
28.5.25	1916	20	1·7	33	23
28.5.25	2106	19	1·9	35	19
28.5.25	2112	17	2·1	36	13
28.5.25	2123	19	1·7	32	21
11.6.25	2107½	17	1·9	31	17
11.6.25	2118½	18	1·5	28	24
18.6.25	2030	19	1·5	29	27
18.6.25	2104	21	1·7	35	25
18.6.25	2117½	19	1·5	29	26
18.6.25	2112½	16	1·9	30	15
25.6.25	1843	11·7	1·1	13	15
25.6.25	2102	18	1·4	25	29
25.6.25	2112½	17	1·9	31	23

of sufficient interest to be worked out where no others are available. This has, therefore, been done in Table VI. The values of $\theta_{max.}$ are here obtained in the same way as those in Table III, for it can be shown in these cases that if a limiting value $Z_{max.}$ be assigned to Z, and if $Z_1 < Z_0$,

$$\sin\theta < \frac{Z_{max.} - Z_0}{\beta - \beta_0},$$

where β is obtained from $Z_{max.}$ by using β/Z as obtained from the curves.

(10.)—*Calculation of the Ratio of Intensity of the Downcoming Wave to that of the Direct Wave.* E_1/E_0.

This ratio, in the case of vertically polarised waves, is obtained from the Z curves at any instant provided the angle of incidence is known at that instant. For at the

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peak values, where we assume the two waves are in phase, we have seen from Section 5, equation (18),

$$\frac{E_1}{E_0} = \left| \frac{Z - Z_0}{Z_0 (1 + \rho_v) \sin \theta} \right| . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Values for ρ_v obtained from equation (21) are given in Table VII.

In the case of the horizontally polarised waves, the ratio of intensities is obtained from the value of α at an instant when θ is known. In order to obtain α, the value of Z must be found at a given value of α/Z. The ratio E_1/E_0 is thus calculated according to equation (20).

The values of ρ_h for various values of θ are also given in Table VII.

TABLE VII.—Values of Reflection Ratio of the Earth's Surface for Vertically and Horizontally Polarised Waves for Various Angles of Incidence.

($\lambda = 386$ metres ; $\sigma = 10^8$ E.S.U.)

θ degrees.	ρ_v.	ρ_h.
0	0·92	0·92
10	0·92	0·92
30	0·91	0·93
45	0·89	0·93
60	0·84	0·97
70	0·78	0·98
80	0·62	0·99

The values of the intensity ratio have been worked out for the various instants recorded in Tables IV and V, and are inserted in the last column of those Tables. It is to be noticed from those Tables that the extreme values of E_1/E_0 are of the same order for both the vertically and horizontally polarised waves.

(11).—*Discussion of Results and Conclusions.*

In this section it will be as well to review briefly the experiments described in this paper, in order to obtain a clear idea of the position of the problem at the time of writing (September, 1925). In the first place the failure of the methods of measuring the directions of the electric and magnetic forces to give positive evidence of the receipt of downcoming wireless waves at the earth's surface has been overcome by working on shorter wave-lengths. Operating on wave-lengths in the neighbourhood of 400 metres it has been shown that the directions of the component forces in the arriving waves can vary considerably from their normal daytime values. Such variations begin at times in the proximity of sunset or after total darkness prevails, and accompany the more commonly known variations in signal strength and apparent bearings as observed

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on a direction-finder. The magnitudes of the variations already observed on the two independent sets of apparatus are considered to provide direct evidence of the reception of wireless waves arriving at the earth's surface at a comparatively small angle of incidence (*e.g.*, 30° or less to the vertical); for although it may contribute to the result, the departure of the path of travel of the wave from the great-circle plane between transmitter and receiver is not alone sufficient to account for the results obtained. Further, from the fact that the variations in the direction of the forces are obtained both in the plane of propagation and at right angles thereto, it may be deduced that the downcoming waves can be polarised with the electric force in either a vertical or horizontal plane.

By combining the above measurements with simultaneous observations of the strength of the vertical electric force of the arriving waves, a means is provided of determining the angle of incidence of both the vertically and the horizontally polarised components of the downcoming waves. Typical results are given in the paper of such determinations, which have been carried out at Slough on the transmissions from the Bournemouth Station, and these show that the resulting angles of incidence can have any value between the limits of 16° and 34°.

A third method of obtaining direct evidence of the existence of downcoming waves at night, and of measuring their angle of incidence, is based upon the fact that in such cases the resultant electric and magnetic forces at the earth's surface are not equal, as they would be for waves travelling horizontally. The results of such measurements carried out simultaneously with those mentioned above show that the angle of incidence may have any value from 13° to 29°. It must be remembered, in considering these results, first that in the experiments all the quantities under observation were varying over large ranges every few seconds, and secondly, that there is no reason to suppose that only a simple wave at a definite and steady angle of incidence is coming down. Indeed, evidence was provided among the observations of the reception of two or more downcoming waves at varying angles, and it is evident that in such cases only a sort of weighted mean angle of incidence can be measured. Bearing in mind, then, that the numerical results represent at least the order of the angle of incidence, let us consider the limiting values of 13° and 34°.

If the last value represents the case of a simple wave which has been symmetrically reflected or deflected from the upper atmosphere in its passage from Bournemouth to Slough, the maximum height of this path becomes 55 miles or 88 kilometres. This is in good agreement with recent estimates of the height of the ionised deflecting layer for wave-lengths of the order employed; for example, in experiments of APPLETON and BARNETT* on the same wave-length, a value of 80–90 km. was obtained for this height, while BREIT and TUVE†, operating on a wave-length of 71·3 metres, have obtained

* APPLETON and BARNETT, 'Nature,' Vol. 115, p. 333, 1925; also 'Proc. Roy. Soc.,' A, Vol. 109, p. 621, 1925.

† BREIT and TUVE, 'Nature,' Vol. 116, p. 357, 1925.

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experimental evidence for a deflecting layer at a height of 80 km. With regard to the other small angles of incidence recorded above, it is evident that if these also refer to a simple wave propagated in the above manner, the height at which the deflection occurred must be greater; it would have to be about three times the above values in order to secure an angle of incidence of 13° or less. Another manner of viewing the matter, however, takes account of the multiple reflection of waves from the upper atmosphere and the earth alternately. The values in Table VII show that the earth is a relatively good reflector for wave-lengths and angles of incidence of the order under consideration; and waves which have been deflected two, three and four times from the upper atmosphere would give angles of incidence of about 19°, 13°, and 10° respectively. How important such multiply reflected waves are depends on their intensities relative to that of the main wave, and is one of the subjects that remains for future investigation.

Having obtained the angle of incidence, the experimental results also furnish data by which the relative intensities of the downcoming and direct waves may be determined. Some measurements in the paper have given values varying from 0·3 to 1·4 for this ratio, showing that under the conditions of those experiments the intensity of the downcoming wave can be of the same order as that of the direct wave. Under such conditions the " reflection coefficient " of the upper ionised layer can be estimated if the attenuation of the direct wave along the earth's surface is known. Unfortunately no experimental measurements of this quantity have so far been carried out in this country. By substituting the known value of the earth's conductivity in the theoretical formulæ given by SOMMERFELD, a value of attenuation of the waves travelling from Bournemouth to Slough can be obtained. On such a basis, the " reflection coefficient " of the upper layer can be shown to be about 0·2.

Although in this general investigation a few preliminary observations have already been carried out at Slough on the transmissions from the Birmingham station, there are insufficient definite results to record here. In the light of the experience gained, it is proposed to continue the investigation on the transmissions from Bournemouth and other stations, operating on different wave-lengths and with different directions of transmission. It is hoped thereby to obtain further data and to improve the accuracy of the results so far obtained.

(12).—*Acknowledgments.*

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XVII.

FURTHER MEASUREMENTS ON WIRELESS WAVES RECEIVED FROM THE UPPER ATMOSPHERE

BY

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ABSTRACT.

Continues observations (described in Papers 15 and 16) made at Slough on waves from the Bournemouth station. Waves from London, Newcastle and Birmingham were also observed. The relative values of the vertical electric force Z, the angles of inclination of the electric force, A and B, the angles of inclination of the magnetic force, C, D and F, and the ratio of the horizontal magnetic force to the vertical electric force were measured simultaneously, half-hourly, for four hours at night and for short periods during the day. Each set of observations took five minutes, during which the four instruments were read simultaneously every five seconds. For London, 20 miles from Slough, the variations of signal strength for sunset or night periods were less than 10 per cent. of the day value, while for Newcastle the night signal intensity was as much as twelve times the day value. The Birmingham values were from 25 to 100 per cent. greater at night, with errors in bearing ranging up to 56°. Bournemouth showed excessively large rotations of apparent bearings. The effective height of deflecting layer calculated from Birmingham was 101 km., from Bournemouth 84 km., from Newcastle 105 km. Values of the coefficient of reflection from the Heaviside layer were also obtained.

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I.—*Introduction.*

IN Papers 15 and 16* the authors have described the development of experimental methods of measuring the directions and relative intensities of both the electric and magnetic forces in wireless waves received at the earth's surface from a distant transmitting station. In this work it was seen that the detection of the arrival of waves deflected from the upper atmosphere, and polarised with their electric force in a horizontal plane, was rendered difficult owing to the relatively great reflecting power of the earth resulting from its high conductivity. By a suitable choice of wave-length and careful attention to detail in the design and construction of the apparatus, however, the methods employed enabled measurements to be made on both vertically and horizontally polarised waves. The results of such measurements enabled a direct proof to be given of the fact that the fading of wireless signals on a vertical aerial, and the variations of bearings experienced on the closed-loop type of wireless direction-finder, are due to the reception, respectively, of vertically and horizontally polarised waves deflected from the upper atmosphere in their passage from the transmitter to the receiver. On arrival at the receiver, these indirect or atmospheric waves interfere with the direct or ground waves, in a manner determined by their relative magnitude and phase, and produce the intensity and apparent directional variations mentioned above. The results of such interference phenomena have been investigated experimentally by APPLETON and BARNETT† and by HOLLINGWORTH.‡ In a more recent publication, the present authors§ have provided experimental evidence showing that the path of the indirect waves is confined to the great circle plane between the transmitter and receiver.

The measurements of the quantities in the received waves, as previously described by the authors, were confined to observations on the transmissions from the Bournemouth broadcasting station over a short period. The object of the present paper is

* SMITH-ROSE and BARFIELD, 'N.P.L. Collected Researches,' Vol. 21, Papers 15 and 16, pp. 317, 333; or 'Proc. Roy. Soc.,' A, Vol. 107, p. 587, 1925, and Vol. 110, p. 580, 1926.

† APPLETON and BARNETT, 'Proc. Roy. Soc.,' A, Vol. 109, p. 621, 1925; Vol. 113, p. 450, 1926.

‡ HOLLINGWORTH, 'N.P.L. Collected Researches,' Vol. 21, Paper 20, p. 407; or 'J. Inst. Elec. Eng.,' Vol. 64, p. 579, 1926.

§ SMITH-ROSE and BARFIELD, 'N.P.L. Collected Researches,' Vol. 21, Paper 18, p. 379; or 'J. Inst. Elec. Eng.,' Vol. 64, p. 831, 1926.

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to describe the continuation of these measurements and their extension to the transmissions from other stations.

II.—*Procedure of the Experiments.*

These measurements were carried out during the six months from July to December, 1925, the transmissions from the broadcasting stations at Bournemouth, Birmingham, London and Newcastle being utilised for the purpose. As previously described, the experiments comprised the carrying out of simultaneous observations on the following four sets of apparatus :—

(*a*) *Signal strength apparatus*, measuring the relative strength of the vertical electric force (Z).*

(*b*) *Hertzian rod apparatus*, measuring the angles of inclination of the electric force (A and B).

(*c*) *Tilting coil apparatus*, measuring the angles of inclination of the magnetic force (C, D and F).

(*d*) *Loop-aerial apparatus*, measuring the ratio of the horizontal magnetic to the vertical electric force α/Z and β/Z.

Observations were made simultaneously on all these instruments over a five-minute period, which was repeated half-hourly for four hours. This programme was repeated on several occasions for each of the transmitting stations observed. Owing to the rapid variation in all the quantities measured, it was found necessary to make an observation every five seconds during the five-minute periods, and this naturally required accurate synchronisation of the watches of the four independent observers.

It will be seen that the four sets of apparatus provided measurements of eight quantities relating to the forces in the arriving wave-fronts, and the whole series of measurements provided a considerable mass of data, which has been systematically reduced in the following manner.

III.—*Treatment of Data Obtained.*

The eight quantities measured during this investigation can best be defined by means of the system of rectangular co-ordinates previously employed* ; they are as follows :—

(*a*) The strength of the vertical electric field = Z.
(*b*) Angle A = $\tan^{-1}$ (Y/Z).
(*c*) Angle B = $\tan^{-1}$ (X/Z).
(*d*) Angle C = $\tan^{-1}$ (α/β).
(*e*) Angle D = $\tan^{-1}$ (γ/β).
(*f*) Angle F = $\tan^{-1}$ (γ/α).
(*g*) Ratio α/Z.
(*h*) Ratio β/Z.

* The notation here used is the same as that employed in Paper 16 (see p. 344).

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In addition to these directly observed quantities it was found desirable to derive two others, viz. :—

(*j*) The component of the horizontal magnetic force, α, derived from (*a*) and (*g*).
(*k*) The component of the horizontal magnetic force, β, derived from (*a*) and (*h*).

An inspection of the graphs of the above quantities showed immediately the general nature of their variations, the times at which each variation began and passed through its maximum value, and the presence or absence of downcoming waves polarised with the electric force either vertical or horizontal.

At selected instants at which the electric field (Z) passed through a maximum, the angle of incidence of the downcoming waves was calculated from the formula

$$\sin \theta = \left| \frac{Z - Z_0}{\beta - \beta_0} \right|,$$

using the curves of Z and β, the day values of which are Z_0 and β_0 respectively. A similar procedure was adopted for peak values of β. Owing to the limitations imposed by the instrumental accuracy in observing Z and β, the above calculations of the angle of incidence were only carried out for cases in which the values of Z or β differed by more than 30 per cent. from the day values as an arbitrarily chosen limit. Of the values of θ thus obtained for each five-minute run, the extremes and means were tabulated with those obtained from the other five-minute runs.

To obtain the angle of incidence of horizontally polarised waves, the relation

$$\sin \theta = \sqrt{K'} \tan F$$

is employed, where $\sqrt{K'} = \sqrt{2\sigma/n}$ = refractive index of ground. Here again, owing to the falling off in instrumental accuracy for weak downcoming waves, it was necessary to impose a limiting condition, which, in this case, was arbitrarily decided as

$$\alpha > \tfrac{1}{2}Z_1 \quad \text{or} \quad > \tfrac{1}{2}\beta_1.$$

The values of θ calculated for values of F, at times when α satisfied this condition, were treated in a similar manner to that mentioned above, the extremes and means being tabulated.

The relative intensity of the vertically polarised component of the downcoming waves is calculable from the relationship

$$\frac{E_1}{E_0} = \frac{1}{(1 + \rho_v)} \cdot \frac{\beta_1}{Z_0}.$$

In a similar manner, for the horizontally polarised component of the downcoming

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waves, the following expression is used for obtaining its intensity (E_1) relative to that of the direct wave (E_0) :—

$$\frac{E_1}{E_0} = \frac{\alpha}{(1 + \rho_h) \cos \theta \therefore Z_0}.$$

In this latter case, therefore, a value of E_1/E_0 can be obtained for each of the derived values of θ.

The extreme and mean values of the intensities and angles of incidence of both components of the downcoming waves, during certain five-minute periods of observation, are entered in the Tables II and III below for certain measurements made on the transmissions from Bournemouth and Newcastle. In the following section is given a detailed discussion of the results obtained from the measurements on the different transmitting stations.

IV.—*Discussion of Results.*

(*a*) *General.*—In Table I is given a summary of the D.F. bearing and signal strength observations made during these experiments on the transmissions from four broadcasting stations. Several general deductions may be made from these results on the direct and downcoming waves received from the various stations. For example, in the case of London's transmissions at a distance of 20 miles, the variations of signal strength during any sunset or night period were less than 10 per cent. of the day value, while the corresponding variations in the apparent bearings had a maximum amplitude of 4°. It is, therefore, to be concluded that the intensity of the downcoming waves from this station is very small compared with that of the direct wave along the ground. On the other hand, the normal day signal intensity from Newcastle is 1·6 units, and this rises at night to as much as 18·4, or nearly twelve times as great. This indicates that the direct wave from this station is comparatively weak and that at night the reception is chiefly due to the downcoming wave. It is seen that the variations of apparent bearings due to the changing polarisation of these latter waves give a maximum error of 49°.

The results obtained on the transmissions from Birmingham occupy an intermediate position between those just discussed. The variations in signal intensity at night indicate that the downcoming waves at night contribute a vertical electric field component, which is from about 25 to 100 per cent. of the normal day value. This variation is accompanied by errors in the apparent bearing of the station ranging up to 56°. A somewhat different case, again, is presented by the results on transmissions from Bournemouth. While the normal day signal intensity is quite appreciable, the variations at night indicate that the downcoming waves can supply a vertical intensity component, which is equal to, and at times greater than, the day value. It is evident that in this case the conditions of wave-length, distance of transmission, etc., are such that the downcoming waves have an intensity which is of the same order as that of the direct waves. As a result, critical interference conditions may occur, giving the

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Transmitting station.	Distance.	True bearing.	Date.	Period of test.	Time of sunset.	Mean day bearing.	**Mean day** value of signal strength Z.	**Maximum** deviations of bearing from mean.	**Maximum** variations of Z from day mean.	Remarks.
	Miles.	Degrees.		G.M.T.	G.M.T.	Degrees.	Arbitrary units.	Degrees.		
Newcastle ...	245	350	12.11.25	1630–2030	1616	351·2	1·6	−21·2 +23·8	+17·2	
,, ...			19.11.25	1600–1930	1606	351·4	1·6	−26·4 +48·6	+18·4	
London ...	18	84	8.10.25	1640–2040	1724	84·0	—	− 4·0 + 3·5	—	Extreme variation of signal strength during these tests was less than 10 per cent. of the day value.
,, ...			15.10.25	1650–2030	1708	85·2	—	− 1·0 + 1·3	—	
,, ...			17.12.25	1730–2030	1552	85·4	—	− 1·4 + 1·1	—	
Birmingham ...	88	319	2.7.25	1450–1600 1930–2130	2020	321·0	7·0	−11·0 +19·0	+ 8·1 − 0·2	*I.e.*, lowest reading 6·8 at night.
,, ...			9.7.25	1450–1600 1845–2125	2016	320·9	8·0	−10·9 + 9·1	+ 7·2	*I.e.*, always above 8 at night.
,, ...			16.7.25	1440–1600 1845–2145	2010	321·4	13·0	− 4·4 + 6·6	+ 3·4 − 3·0	
,, ...			17.9.25	1725–2125	1812	321·4	7·0	−21·4 +22·6	+ 2·6 − 1·8	
,, ...			1.10.25	1445–1545 1725–2125	1739	320·9	6·6	−55·9 +39·1	+ 3·4 − 2·9	
,, ...			5.11.25	1630–2030	1627	321·0	6·0	−21·0 +39·0	+ 5·2 − 1·2	Morse interference.
Bournemouth...	79	228	28.5.25	1450–1540 1845–2125	2002	226·8	12·0	+103·2 −116·8	+10·0 − 8·3	
,, ...			4.6.25	1445–1600 1845–2125	2010	231·6	12·0	+ 38·4 − 36·6	+ 5·0 − 4·5	Fairly quiet night for 6 BM.
,, ...			11.6.25	1445–1600 1845–2125	2016	232·0	12·0	—	+ 8·4 − 2·8	Observing F most of period.
,, ...			18.6.25	1450–1605 1845–2125	2020	232·2	12·0	—	> +10·0 − 3·0	Observing D and F most of period.
,, ...			25.6.25	1445–1605 1845–2125	2021	232·1	12·0	—	> +10·0 − 2·5	Do.
,, ...			24.9.25	1510–1545 1715–2125	1755	231·1	8·0	+ 124 − 161	+ 8·0 − 4·4	
,, ...			29.10.25	1630–2030	1639	232·0	8·0	± 180	+18·0 − 2·5	Complete rotation 2–3 occasions.
,, ...			10.12.25	1540–1940	1551	232·0	—	± 180	—	Do. on three occasions.

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excessively large rotations of apparent bearings which have only been observed on this station.

With this introductory discussion the results may now be considered in more detail under the heading of each transmitting station.

(*b*) *Birmingham.*—The Birmingham transmitting station was observed on six occasions each consisting of a four-hour continuous watch in the evening. It was at once found that the fading and other night effects were not so marked with this station as with Bournemouth. The total downcoming wave was, in fact, found to be only of the order of one-tenth the amplitude of the direct wave, whereas in the case of Bournemouth it averaged one-third. This fact, which is satisfactorily accounted for by the smaller surface attenuation of the direct Birmingham wave compared with the Bournemouth wave, rendered it impossible to make an accurate calculation of the angle of incidence of the downcoming wave by the method outlined above. A rough estimate, however, was made on the same lines, and this showed that the results were consistent with a downcoming wave having an angle of incidence lying between 30° and 40°. The distance from Birmingham to Slough being 88 miles, the mean of the above angles of incidence (35°) gives an effective height of the deflecting ionised layer of 63 miles, or 101 km., for transmissions from this station.

(*c*) *Bournemouth.*—Since the results recorded in the preceding paper* this station has been observed on three occasions, each consisting of four-hour watches in the evening preceded by a day-time run of half an hour to establish normal conditions. The fading and other night effects observed on this station were, as before, very marked. Variations in Z of $\pm$ 50 per cent. to 100 per cent. were quite common, and variations in bearing (angle C) were particularly large—on several occasions a complete rotation of 360 degrees took place. Large forward and sideways tilts of the electric field were also observed.

The angles of incidence and relative intensities of the downcoming waves were measured when possible and are shown in Table II.

The apparent angle of incidence is seen to vary rapidly from minute to minute. The mean angle of incidence—42° for the vertically polarised waves, and 32° for the horizontally polarised waves—will give an approximation to the *mean* angle at which the total downcoming radiation is arriving. The vertically polarised mean is probably too high owing to the tendency of weak downcoming waves to give rise to instrumental error in this direction. For the same reason the horizontally polarised mean should be too low. Hence, we can take the average of the two (37°) for the true angle with considerable confidence. This angle gives 84 km. as the mean height to which the atmospheric waves penetrate, which is in good agreement with the result (88 km.) given in the preceding paper.†

* Smith-Rose and Barfield, 'N.P.L. Collected Researches,' Vol. 21, Paper 16, p. 333; or 'Proc. Roy. Soc.,' A, Vol. 110, p. 580, 1926.

† Paper 16, p. 363.

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TABLE II.—Transmissions from Bournemouth. ($\lambda = 386$ m.)

Date.	G.M.T.	Vertically polarised wave.				Horizontally polarised wave.			
		Angle of incidence.		Intensity.		Angle of incidence.		Intensity.	
		Extremes.	Mean.	Extremes.	Mean.	Extremes.	Mean.	Extremes.	Mean.
		Degrees.	Degrees.			Degrees.	Degrees.		
24.9.25	1845					(0)–58	30	0·23–1·10	0·59
	1900	59–(90)	75	0·20–0·29	0·26				
	1945					26–47	39	0·42–1·44	0·87
	2000	17–49	27	0·29–0·80	0·52				
	2015					26–76	39	0·32–1·26	0·58
	2030	20–38	25	0·35–0·62	0·46				
	2045					(0)–58	29	0·25–0·73	0·42
	2100	20–50	34	0·15–0·26	0·21				
	2115					(0)–47	15	0·25–0·69	0·46
29.10.25	1700	19–37	29	0·30–0·87	0·45				
	1745					16–76	35	0·43–4·90	1·67
	1800	24–48	38	0·21–0·26	0·23				
	1815					16–58	33	0·33–0·77	0·45
	1930	32–47	40	0·20–0·27	0·23				
	1945					47–58	52	0·48–0·61	0·54
	2000	29–(90)	58	0·20–0·44	0·34				
	2015					(3)–47	29	0·28–0·73	0·44

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The ratio of the field strength of the downcoming wave to the direct wave, as measured at the instants at which the angles of incidence were calculated, and, therefore, probably representing the maximum value, is found from Table II to average 0·34 for the vertically polarised and 0·67 for the horizontally polarised wave. Thus the total horizontally polarised wave appears to have about twice the intensity of the total vertically polarised wave. Of the two figures, however, it should be pointed out that the former is the more reliable, since the value for the horizontally polarised wave is the result of a less direct calculation.

Reflection Coefficient of the Heaviside Layer.—If we define the reflection coefficient (ρ) of the Heaviside layer as the ratio of the intensity of the electric field of the wave on leaving the layer to its value on entering, it can be shown that the expression for determining it is

$$\rho = \frac{a \,.\, E_1/E_0}{\sin\theta \,.\, f(\theta)},$$

where

E_1/E_0 is the ratio of downcoming to direct wave observed at the receiver, as recorded in Table II.

a is the surface attenuation factor of the direct wave due to earth resistivity.

θ is the angle of incidence of the downcoming wave.

$f(\theta)$ is the ratio of the field intensity radiated upwards by the transmitter at angle θ to the value radiated along the ground.

In the case of Bournemouth we have

$$E_1/E_0 = 0{\cdot}34 \quad \text{(vertically polarised wave).}$$

The value of a is obtained from SOMMERFELD'S attenuation theory, utilising a value for the effective surface resistance obtained by actual attenuation measurements on a wave-length close to that of Bournemouth and over a considerable part of the same ground. Its value for the particular wave-length, distance and soil condition relating to Bournemouth is approximately 0·04. The angle of incidence (θ) is obtained as in Table II and assumed to be 37°. The value of $f(\theta)$ is based on the assumption that the polar curve of radiation in a vertical plane is the same as that of a vertical aerial, and for $\theta = 37°$ is 0·2. Putting these values in the above equation we obtain $\rho = 0{\cdot}11$ as the value for the reflection coefficient of the Heaviside layer for waves from Bournemouth to Slough.

(*d*) *Newcastle.*—The transmissions from Newcastle were observed on two night periods of four hours each, with suitable day periods for obtaining the necessary day values. The results are shown in Table III.

The angle of incidence of the downcoming wave was found to remain more constant

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TABLE III.—Transmissions from Newcastle. ($\lambda = 410$ m.)

Date.	G.M.T.	Vertically polarised wave.			
		Angle of incidence.		Intensity.	
		Extremes.	Mean.	Extremes.	Mean.
		Degrees.	Degrees.		
12.11.25 ...	1730	64–69	66	2·53–4·18	3·3
	1800	56–65	62	1·58–1·91	1·8
	1830	20–77	52	1·22–2·37	1·9
	1900	13–46	28	0·83–1·84	1·2
	1930	52–90	64	1·64–2·73	2·2
	2000	62–68	65	0·92–1·38	1·2
19.11.25 ...	1630	58–73	63	0·92–1·48	1·2
	1700	58–70	63	1·05–1·15	1·1
	1730	51–90	68	1·25–2·04	1·6
	1800	60–90	68	0·92–1·64	1·2
	1830	66–72	70	2·76–3·95	3·4
	1930	62–90	73	3·26–5·69	4·3

than for Bournemouth. The mean angle for the whole of the observations was 62°. The ratio of indirect to direct wave intensities averaged 2·0.

The intensity of the horizontally polarised component was not great enough to enable calculation of the angle of incidence of the horizontally polarised wave to be made. As already mentioned, however, the variations in apparent bearings recorded in Table I indicate that the intensity of the horizontally polarised downcoming waves was at times very appreciable.

Height of Layer: Reflection Coefficient.—The height of the Heaviside layer calculated from the average angle of incidence given above is 105 km. The reflection coefficient calculated in the same way as for Bournemouth and Birmingham is 0·14.

V.—*Conclusions.*

The method of analysis of the electromagnetic field in the presence of downcoming waves at receiving stations, which was applied to the waves received at Slough from the Bournemouth station, has been extended to deal with two other stations at different distances and on different wave-lengths, and further confirmatory observations have been made on Bournemouth.

From measurements of the mean angle of incidence of the downcoming waves by the loop-aerial method, the height of the layer from the various stations has been worked out. Also, by making use of the results of some experiments on the attenuation

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of the surface wave over land, it has been possible to arrive at more exact values for the coefficient reflection of the Heaviside layer from the three cases dealt with. These results are summarised in Table IV below.

TABLE IV.

Transmitter.	λ.	Height of layer.	Coefficient of reflection of ionised layer.
	m.	km.	
Bournemouth	386	84	0·11
Birmingham	479	101	0·17
Newcastle	410	105	0·14

The preceding paper* contained the results of measurements on downcoming waves received at Slough from Bournemouth during May and June, 1926. In the following Table V these results are compared with the corresponding results in Table II of the present paper, referring to similar measurements made in September and October, 1926.

TABLE V.—Results of Measurements on Transmissions from Bournemouth made in the Summer and Autumn, 1926.

Date of measurement.	Vertically polarised waves.		Horizontally polarised waves.	
	Mean angle of incidence.	Mean intensity.	Mean angle of incidence.	Mean intensity.
	degrees.		degrees.	
18th and 25th June, 1926 ...	21	0·7	29	0·8
24th Sept. and 29th Oct., 1926...	40	0·35	33	0·7

It is interesting to note from this table that while the results of the measurements on the horizontally polarised waves are of the same order in the two cases, the vertically polarised waves in the autumn arrive at a smaller angle of incidence and with half the intensity of the waves received at midsummer. As already explained in this paper, these results probably give the mean value of two or more downcoming waves, and the above change may simply indicate a reduction of the intensity of those waves

* Paper 16, pp. 359, 361.

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which have suffered two deflections at the ionised layer. Alternatively, the results may indicate a real change in the effective height of the ionised layer.

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XVIII.

THE CAUSE AND ELIMINATION OF NIGHT ERRORS IN RADIO DIRECTION-FINDING

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WITH

AN APPENDIX

BY

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ABSTRACT.

The paper describes experiments which have been carried out with a view to obtaining more conclusive evidence as to the causes of the apparent variations in bearings observed under certain conditions on wireless direction-finders. In the course of the experiments the Adcock "four-aerial" direction-finder has been developed, and with its aid it has been shown that the actual deviation in azimuth of wireless waves is practically negligible. These experiments thus constitute a proof that the variable errors observed on closed-coil direction-finders at night are caused by downcoming waves arriving from the upper atmosphere and polarised with the electric force in a horizontal plane. The investigation also indicates the possibility of the Adcock system being developed into a practical direction-finder which is free from night errors, and from those errors associated with observations on aircraft transmissions made at a ground direction-finding station.

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THE CAUSE AND ELIMINATION OF NIGHT ERRORS IN RADIO DIRECTION-FINDING.

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I.—*Object of the Present Paper.*

The object of a large proportion of the investigations carried out by the present authors for the Radio Research Board during the past few years has been to ascertain definitely the cause of the apparent variations in bearings experienced on closed-coil direction-finding apparatus during the " night " periods. Some of these investigations have led to the development of methods of distinguishing between the electric and magnetic components of wireless waves, and of the separate measurement of their intensity and direction. This portion of the work has recently led to a definite proof of the existence of downcoming waves at the earth's surface with components of suitable polarisation and sufficient intensity to account for a large proportion of the night errors experienced.*

In this manner the theory originally advanced by T. L. Eckersley† in 1921 to account for the apparent variations in the direction of arrival of wireless waves has been adequately confirmed; and it is now evident that these variations are caused by the action of the horizontal components of electric force in the downcoming waves on the horizontal parts of the direction-finding loops. A good discussion of these effects and the associated phenomena is given in Keen's book on the subject of wireless direction-finding.‡

It is generally assumed that the downcoming waves have travelled *via* the upper regions of the earth's atmosphere without deviating laterally from the great-circle plane through the transmitter and receiver. This assumption has, however, not been sufficiently well justified in the past; and the presence of the downcoming waves has hitherto made it difficult to prove whether the direct waves travelling along the earth's surface have not also suffered any lateral deviation. Another portion of the authors' investigations has therefore been made to ascertain to what extent, if any, such lateral deviation exists. The problem is one of more than pure scientific interest; for it is evident that, if directional receiving apparatus can be devised which shows the absence

* R. L. Smith-Rose and Barfield, 'N.P.L. Collected Researches,' Vol. 21, Papers 16 and 17, pp. 333 and 365; or 'Proc. Roy. Soc.,' A, Vol. 110, p. 580, 1926.

† T. L. Eckersley, 'Radio Review,' Vol. 2, pp. 60, 231, 1921.

‡ J. Keen, 'Direction and Position Finding by Wireless,' p. 164, Wireless Press, London, 1922.

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of lateral deviation, we are at once provided with a direction-finder which is free from the variable night errors that place such a serious limitation on closed-coil direction-finders. The present paper describes the experiments which have been carried out with these objects during the years 1925 and 1926.

II.—*Suggestions and Attempts by Previous Workers.*

The problem involves essentially the production of an apparatus which will measure the horizontal component of the direction of arrival of wireless waves, irrespective of their angle of incidence at the earth's surface or their state of polarisation.

Three methods of attacking this problem have previously been described. In a patent filed in 1919, F. ADCOCK* described a direction-finding arrangement in which the aerials were mounted and connected in such a manner as to ensure that only the vertical parts were acted upon by the arriving waves. In this way he proposed to eliminate the errors which are caused by the effect of the horizontal electric field on the receiving system. This arrangement consisted essentially of four vertical aerials spaced at the corners of a square ; the aerials at the extremities of a diagonal formed a pair and were connected in opposition to the field coils of a radiogonimeter. Provided that the horizontal members of the system have no effective E.M.F. induced in them, it is clear that the current in the field coils is due entirely to the phase difference of the E.M.F.'s induced by the vertical component of the electric force of the arriving waves. This current will thus depend only on the horizontal direction of arrival of the waves relative to the system.

In the paper already mentioned, T. L. ECKERSLEY† suggested that any lateral deviation of wireless waves from the great-circle plane through transmitter and receiver could be measured by employing a receiver in which each of the ordinary closed loops of the Bellini-Tosi arrangement is replaced by two loops in the same plane, connected together in series and so that their E.M.F.'s oppose each other. With such an arrangement the effective E.M.F. induced therein will be entirely due to the phase difference between the individual E.M.F.'s in the two loops resulting from their spacing. It is then evident that a wave arriving in a direction at right angles to the plane of the loops will produce no resultant E.M.F. in the circuit, whatever may be the angle of incidence or state of polarisation of the arriving waves.

In 1921, G. M. WRIGHT and S. B. SMITH‡ published an account of experiments carried out with a direction-finder connected in such a manner as to give the heart-shaped polar diagram for reception. These experiments showed that, under conditions when large variations in apparent bearings were obtained on the ordinary direction-finder, the position of the heart-shape minimum remained constant, although the actual shape of the polar

* F. ADCOCK, British Patent 130,490 (1919).

† T. L. ECKERSLEY, *loc. cit.*, p. 239.

‡ G. M. WRIGHT and S. B. SMITH, ' Radio Review,' Vol. 2, p. 394, 1921.

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reception curve altered considerably. The agreement of the experimental results with those predicted from ECKERSLEY's theory served as a useful confirmation of this theory. The accuracy of such experiments, however, largely depends upon the purity of the minimum, and, as WRIGHT and SMITH pointed out in their paper, the heart-shaped minimum only remains pure when the downcoming waves are arriving at a reasonably small angle to the horizontal (*i.e.*, a large angle of incidence). In some of the authors' recent experiments, however, angles of incidence as small as 16° to the vertical have been measured,* and the behaviour of the heart-shaped diagram under such conditions is a matter for further investigation.

As far as the authors are aware, no account of the practical working of either of the above systems as direction-finders has yet been published, although the heart-shape circuit arrangement is now in common use for the determination of the " sense " of wireless bearings. Quite recently, however, a directional receiving system comprising two spaced loops mounted in a common vertical plane has been described by H. T. FRIIS.† This arrangement as been developed with a view to obtaining directional selectivity in ordinary wireless reception, and not as an application to direction-finding.

III.—*Early Experiments at the Radio Research Station, Slough.*

(*a*) *Double Loop Method.*—In the first experiments, a modification of ECKERSLEY'S suggested method was employed. Instead of using two pairs of large double loops at right angles, a single pair of small loops was constructed and arranged in a vertical plane so as to be rotatable about a vertical axis.

The E.M.F. induced in such a system by a single wave is given by the expression :—

$$e = \mathrm{K} \sin \frac{2\pi d \sin \theta}{\lambda} \cos \psi \, (\alpha \sin \psi + \beta \cos \psi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

K is a constant for loops of a given area,

d = distance between the centres of the loops (assumed small compared with the wave-length λ),

ψ = angle between plane of loops and horizontal direction of waves,

θ = angle of incidence of the waves at the earth's surface,

α = horizontal component of magnetic field of waves in great-circle plane through transmitter and receiver,

β = horizontal component of magnetic field of wave perpendicular to α.

The above expression for the induced E.M.F. becomes zero when $\psi = 90°$, so that there is a zero or " minimum " position of the system when the plane of the coils is at

* SMITH-ROSE and BARFIELD, this Volume, Paper 16, p. 363.

† H. T. FRIIS, ' Proc. Inst. Radio Eng.,' Vol. 13, p. 685, 1925.

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right angles to the horizontal direction of arrival of the waves, whatever may be their state of polarisation or angle of incidence. It should be noted that this is not necessarily the only minimum which will be experienced in the rotation of the coil from 0° to 180°, since it is quite possible to have a second minimum at the value of ψ for which $\tan \psi = -\beta/\alpha$.

For simplicity of construction in the practical arrangement, the loops were made triangular in shape and consisted of a single winding in the form of a figure 8, as depicted in Fig. 1. These loops were erected in a hut which was suitably screened for the elimination of antenna effect, and the remainder of the receiving circuits and the amplifiers were adequately screened from the direct effects of the arriving waves. When the arrangement was tested it was found to work satisfactorily on near-by stations; but owing to the fact that its reception factor is differential in nature, it was not sufficiently sensitive for operation on transmissions from the more distant stations. Hence it was not possible to use the apparatus under the conditions which give rise to the usual

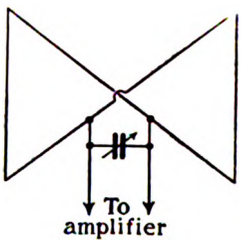

FIG. 1.

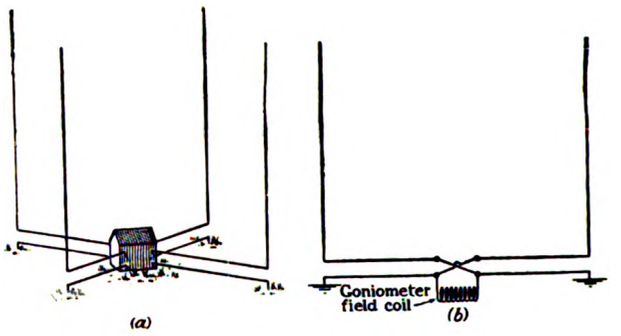

FIG. 2.

(*a*) Sketch of System. (*b*) Elevation of Single Pair of Aerials.

night phenomena in radio direction-finding. In the absence of a much more sensitive amplifier, the only methods of increasing the strength of the received signals would be either to increase the distance between the loops, or to adopt ECKERSLEY'S suggestion of employing large fixed loops on the Bellini-Tosi plan. In regard to the last alternative, it is seen from equation (1) that the E.M.F. is not proportional to $\cos \psi$, and as the operation of the Bellini-Tosi radio-goniometer depends upon such a cosine law, this method cannot be adopted for the present case. Up to the present time no experiments have been made with loops separated by a greater distance, for, as will be seen below, the use of ADCOCK'S method has been found to give more successful results.

(*b*) *Adcock's System.*—The simplest form which this system can take is probably that shown in Fig. 2, where each of the four aerials is earthed at a point vertically beneath it, but the earth lead is taken horizontally to the receiving hut at the centre of the system and back again. In Fig. 2 (*b*) only one pair of aerials is depicted to show the method of connecting the aerials in opposition through the field coil of the radio-goniometer. Such a mode of connexion should have the effect of rendering any E.M.F.'s

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induced in the horizontal members of the system by the arriving waves equal and opposite with regard to the circuit " aerial-field-coil-earth," so that the net E.M.F. induced in this circuit is that due to the phase difference of the individual E.M.F.'s induced in the two vertical members. This E.M.F. is given by the equation

$$e = \mathrm{KZ}\,\frac{2\pi d}{\lambda}\cos\psi, \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

where

K is a constant for a given aerial system,
Z = vertical electric force,
d = distance between the aerials,
ψ = angle between direction of travel of the wave and the line joining the aerials.

Since the E.M.F. is proportional to cos ψ, it is clear that two pairs of such aerials may be used with a radio-goniometer in exactly the same manner that this instrument is used with the fixed loops of the Bellini-Tosi direction-finding system.

Experiments carried out with the above arrangement, as in Fig. 2, showed that, due probably to stray capacities to earth, the horizontal E.M.F.'s induced in the system were not balanced out, and gave rise to serious errors.

IV.—*Attempted Improvement of Adcock's System.*

With a view to overcoming the difficulty just mentioned, an attempt was next made to screen the whole of the horizontal portion of the system from the effects of the arriving waves. Previous experience had shown that in order to screen an aerial adequately, it was necessary for the screen to project beyond the aerial by a considerable length.

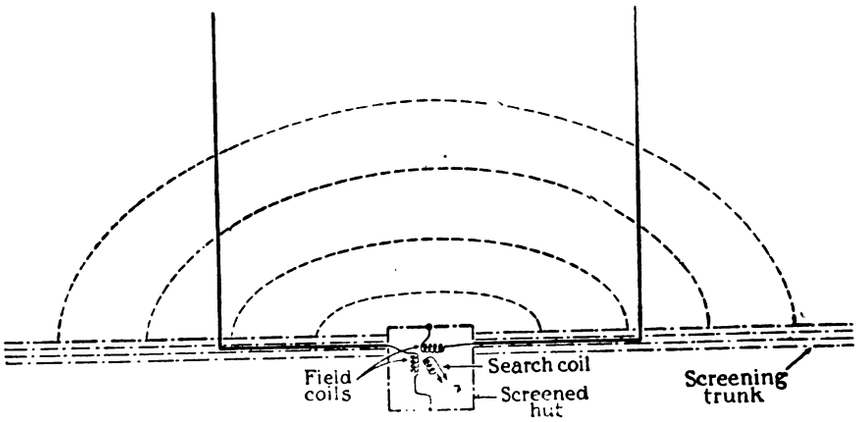

FIG. 3.

The system finally adopted for the present case is depicted in Fig. 3, which shows one pair of aerials with the horizontal members contained within a screening trunk formed of a number of parallel wires. At their inner ends these wires were joined to the metallic

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lining of a screened hut which contained the whole of the receiving apparatus and the operator. Tests of the effectiveness of this screening arrangement were made by measuring the directional properties of the four individual aerials. These measurements unfortunately showed that the attempted screening of the horizontal portion of the aerials had been unsuccessful. The explanation of this result is probably that, whilst the screen effectively prevented any E.M.F. being induced directly in the horizontal members of the aerial system, the vertical component of the secondary electric field resulting from the currents in the screen itself, induced E.M.F.'s in the vertical portions of the aerials (see Fig. 3). Since the currents in the screen are produced by the horizontal component of the electric force in the arriving wave, the condition that this component shall have no effect on the system is not fulfilled.

V.—*Form of Adcock's System Finally Adopted.*

It was now decided to revert to the alternative arrangement described by Adcock* in which the vertical aerials are comprised of complete Hertzian oscillators with leads taken horizontally from their mid-points to the receiving apparatus in the screened hut. In order to maintain symmetry, this, of course, necessitates the raising of the hut with the apparatus and operator to the level of the centre of the aerial system some 20 feet above the ground. For the support of the aerial wires, four poles were available, 44 feet high, situated at the corners of a 20-foot square. The receiving hut was placed at the centre of this system, being suspended by insulated steel ropes from cross beams attached to the poles, to reduce to a minimum its capacity to earth or to other structures. The arrangement is shown diagrammatically in Fig. 4, the circuit being as given in Fig. 5. By this reduction of the capacity to earth of the various parts of the system,

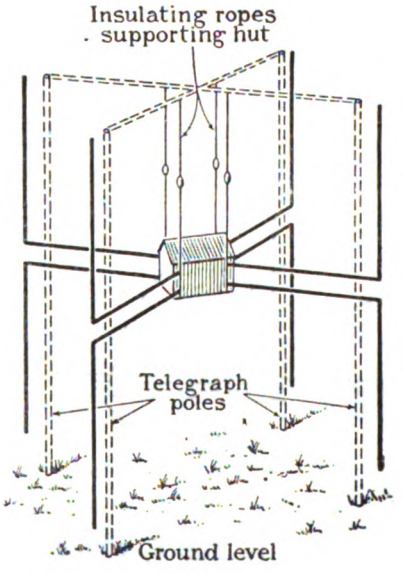

Fig. 4.

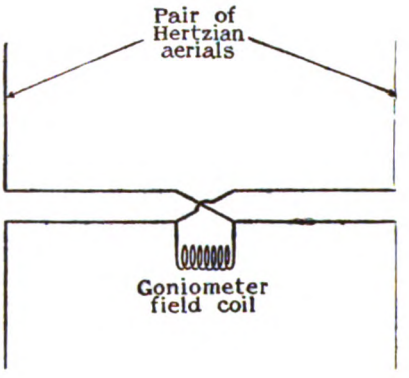

Fig. 5.

* F. Adcock, *loc. cit.*

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it was anticipated that the E.M.F.'s in the horizontal members would completely balance each other, and thus the system would be rendered free from the effects of any horizontal electric fields. The fulfilment of this condition was verified, first by measuring the directional properties of the individual aerials as formerly, and secondly by comparing the induced currents in the upper and lower halves of the same aerial. This last test brought to light a superiority of 2 or 3 per cent. of the lower half over the upper half of each aerial, due probably to the difference in their capacities to earth. Although the error introduced by this asymmetry was very small, it could probably have been eliminated by suitably shortening the lower part of the aerial to make the two halves electrically equal. Since the whole method depends upon the assumption that the four receiving aerials are identical, it is important to observe that other tests showed that the receiving properties of the aerials were not materially influenced by slight differences or irregularities in the neighbouring supporting structures.

VI.—*Calibration of the System.*

The apparatus was now calibrated for direction-finding by taking observations of the apparent bearings of stations transmitting within the broadcasting band of wavelengths, under conditions which gave freedom from night errors. In this manner it was found that the apparatus was subject to a permanent directional error of the same order as that experienced on the closed-coil direction-finders in use in the same field, due to the local surroundings.

VII.—*Determination of Lateral Deviation of Wireless Waves under Night Conditions.*

In order to determine to what extent wireless waves can be laterally deviated from the great-circle plane connecting the transmitter and receiver, it was decided to make observations on some transmissions which were known to give rise to serious night variations on an ordinary direction-finder. Accordingly a series of tests was carried out in which the apparent bearing of a transmitting station was observed simultaneously on the above apparatus and on an ordinary single-coil direction-finder installed in another hut at about 100 yards distance. In this manner observations were taken on the transmissions from Cardiff (353 m.), London (365 m.), Bournemouth (386 m.), Newcastle (404 m.), and Birmingham (475 m.). The most serious night variations were obtained on Bournemouth and Cardiff, and Figs. 6 to 9 show graphically some of the results obtained in the simultaneous observations on these stations. Figs. 6 and 7 refer to the results obtained on Bournemouth on two different evenings; and these show very strikingly that the errors obtained on the Adcock system are almost negligible compared with those obtained on the single-coil direction-finder. Indeed, on several occasions the latter system showed a rotation of the apparent bearing through more than 360°, whilst on the Adcock system the extreme variation in either of the tests was 14°. From

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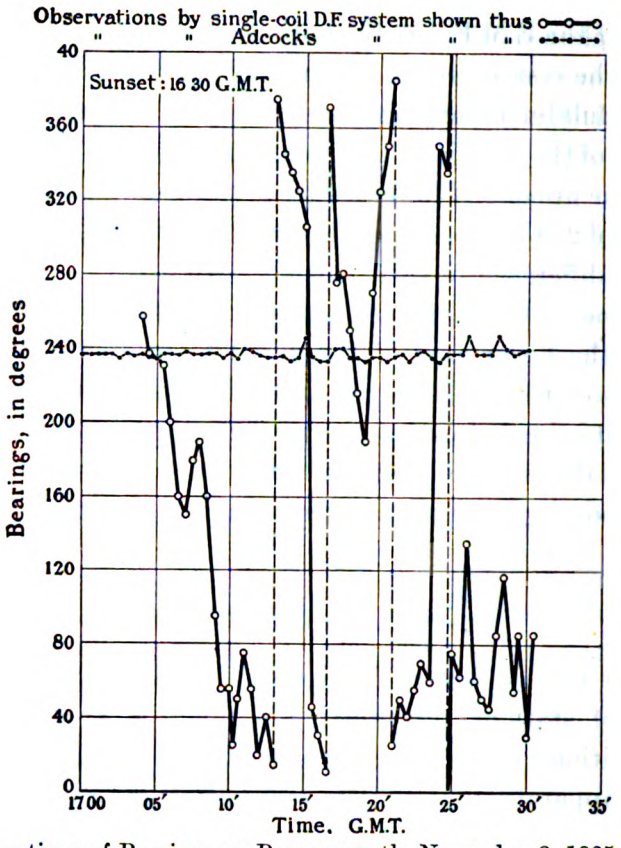

FIG. 6. Observations of Bearings on Bournemouth, November 3, 1925 ($\lambda = 386$ m.).

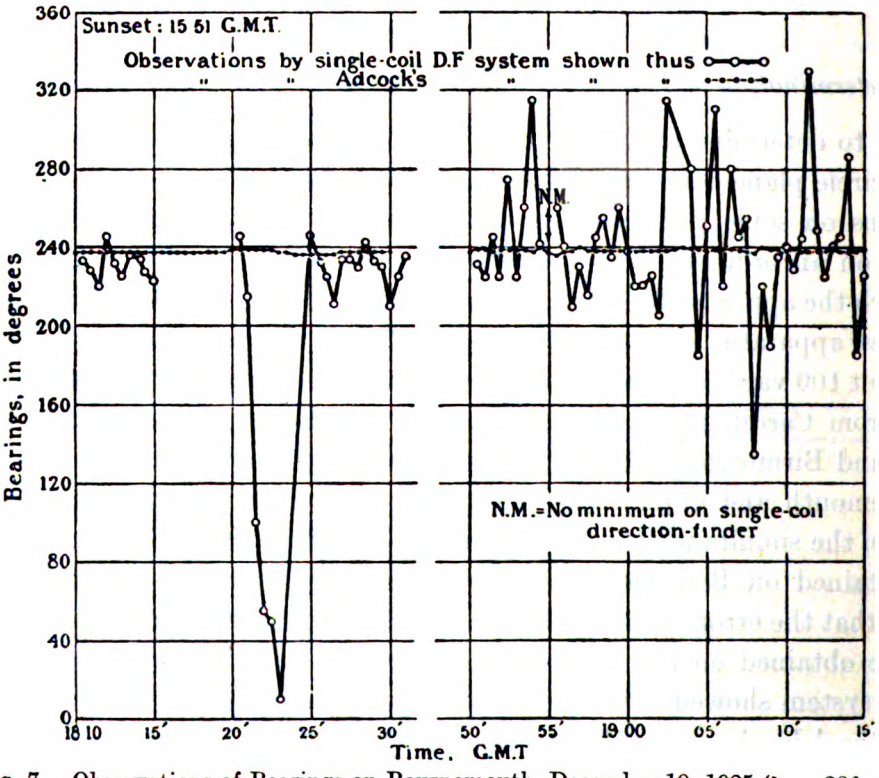

FIG. 7. Observations of Bearings on Bournemouth, December 10, 1925 ($\lambda = 386$ m.).

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the results on Cardiff plotted in Fig. 8 the same effect is observed, although the variations are less violent. On one occasion the bearing on the single-coil direction-finder rotated through 175°, whilst the extreme variation on the other system was about 7°. In the

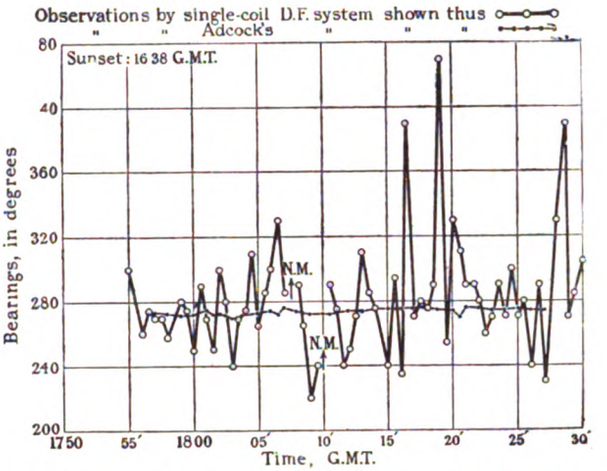

FIG. 8. Observations of Bearings on Cardiff, January 26, 1926 ($\lambda = 353$ m.).

further observations on Cardiff, plotted to a larger scale in Fig. 9, it is seen that when the normal direction-finding variation was comparatively small (14°), the extreme variation on the other set was 1°, which is of the same order as the limiting accuracy of the present apparatus.

It is to be concluded from these results, therefore, that lateral deviation plays a negligible part in producing the large and variable errors in apparent bearings which

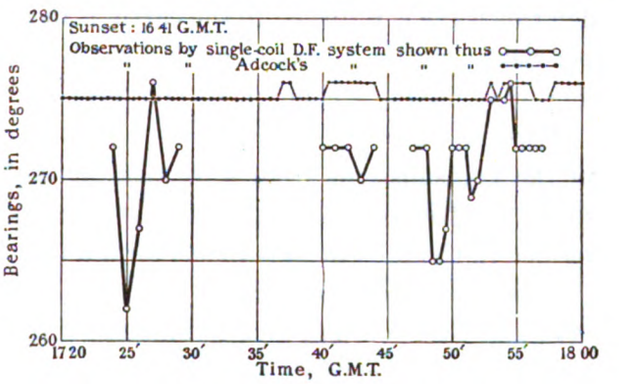

FIG. 9. Observations of Bearings on Cardiff, January 28, 1926 ($\lambda = 353$ m.).

are obtained at night on the present type of closed-coil direction-finding set; and that, therefore, these errors are entirely caused by the arrival of downcoming waves polarised with the electric force horizontal. We have, therefore, in this investigation a further proof of the existence of these downcoming waves which supplements and confirms those given in previous papers by APPLETON and BARNETT* and by the present authors.†

* APPLETON and BARNETT, 'Proc. Roy. Soc.,' A, Vol. 109, p. 621, 1925.

† SMITH-ROSE and BARFIELD, this Volume, Papers 16 and 17, pp. 333 and 365.

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The slight residual variable errors noticed as having been recorded on the "four aerial" system may, it will be seen, amount to $\pm 7°$. These may be caused by a slight defect in the apparatus due to imperfect balancing, by which the rotated downcoming wave is able to produce some slight current in the goniometer, or it may be actually a residual amount of lateral deviation. Until the point is settled by improvement of the apparatus it will be best to assume that it is due to the latter cause. The important point is, however, that even so, an error of $\pm 7°$ due to lateral deviation almost certainly does not imply that the waves are deviating to this extent from the great-circle path between transmitter and receiver, for it may be shown that, given two waves arriving with varying relative phase and with direction of travel in vertical planes separated by an angle ψ, the total maximum apparent variation in the resultant direction will greatly exceed ψ.* The conditions for this occurrence are simply that the two waves shall be approximately equal in amplitude and nearly 180° out of phase. It then appears extremely probable that the apparent variation of $\pm 7°$ recorded is an indication of a much smaller real lateral deviation.

It may further be pointed out that a slight lateral difference in path between the horizontal and downcoming wave is almost certain to exist, for the horizontal wave on land is nearly always slightly deviated by local irregularities of the earth's surface (trees, etc.), whilst the downcoming wave on the other hand will not be influenced by these causes, at any rate to the same extent, and thus the two will probably not arrive exactly in the same vertical plane.

It is to be observed that, since there is no departure of the downcoming wave from the great-circle plane connecting transmitter and receiver, it is unlikely that the upper ionised layer, from which this wave is returned to the earth, is inclined to the horizontal. Such a tilt of the upper deflecting layer was suggested by ECKERSLEY† as a possible cause of the downcoming wave having a component polarised with the electric force in the horizontal plane. Further, since it has previously been shown that directional errors are obtained on the transmissions from a vertical aerial,‡ it is evidently unnecessary for the horizontal electric force to be present in the radiation from the source. The investigations described in the present paper thus form a useful confirmation of the magneto-ionic theory of wave propagation which has been recently introduced by APPLETON and BARNETT§ and by NICHOLLS and SCHELLENG,|| in which the rotation of the plane of polarisation of the downcoming waves is ascribed to the influence of the earth's magnetic field on the motions of the ions in the upper atmosphere.

* See, for example, T. HEILIGTAG, 'Jahrbuch Drahtlosen Telegraphie,' Vol. 21, p. 77, 1923.

† T. L. ECKERSLEY, *loc. cit.*, p. 239.

‡ R. L. SMITH-ROSE, 'J. Inst. Elec. Eng.,' Vol. 62, p. 957, 1924.

§ APPLETON and BARNETT, 'Electrician,' Vol. 94, p. 398, 1925.

|| NICHOLLS and SCHELLENG, 'Bell System Tech. J.,' Vol. 4, p. 215, 1925.

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VIII.—*Application of the System as a Direction-finder Free from Night Errors.*

In the experiments described above it has been demonstrated that ADCOCK's four-aerial directional receiving system is successful in discriminating between laterally deviated waves and horizontally polarised downcoming waves. It is further shown that even during the times when the worst variations of bearings are obtained on the closed-loop direction-finding systems, the amount of actual lateral deviation of the arriving waves is very small. It therefore follows that such a system may be used as a direction-finder which gives the true direction of the great-circle plane of arrival of wireless waves whatever may be their state of polarisation, or their angle of incidence relative to the earth's surface. The system should, therefore, have important applications as an accurate direction-finder under night conditions, or for use in observing on aircraft at high angles of elevation, when the ordinary closed-loop direction-finder is subject to such large errors as to make it practically useless. For example, BALDUS and BUCHWALD* have shown that the closed-coil direction-finder on the ground can give errors of as much as 60° in the bearings of aircraft.

In considering what will be the limiting accuracy of such a system, it must be remembered that the variable error of $\pm$ 7° mentioned above was obtained under what are probably the worst conditions that can prevail in wireless direction-finding on medium wave-lengths. For it is worthy of note that the variations of apparent bearing observed at Slough on the transmissions from Bournemouth are very much greater in both frequency and magnitude than any hitherto experienced by the authors during the past five years on all wave-lengths from 450 to 12,000 m. Under more favourable conditions, such as obtained in the case of the results taken on Cardiff, the maximum error experienced was only 2° or 3°, and it is considered that even this small amount can be reduced by a development of the system on a more favourable site. Also, on the longer wave-lengths allocated for commercial direction-finding the system will be subject to a still smaller error, and the instrumental difficulties will be somewhat decreased.

It is of course realised that in the practical development of the system it would be preferable to have the receiving apparatus, together with the operator, located on or near the ground, and not at the centre of the aerial system as represented in Fig. 4. Several methods of overcoming the difficulties mentioned in Section III (*b*) have already occurred to the authors, and these will be tested experimentally as soon as possible. A consideration of the case also shows that the system may be provided with the sense-finding modification which is now a usual attachment to all direction-finders.

IX.—*Conclusion.*

The experiments have shown that by paying careful attention to symmetry of construction, the system of direction-finding proposed by ADCOCK can be made to work with a considerable amount of success.

* BALDUS and BUCHWALD, 'Jahrbuch Drahtlosen Telegraphie,' Vol. 15, p. 214, 1920.

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It has been employed as a means of discriminating between laterally deviated and horizontally polarised downcoming waves, demonstrating within the limits of experimental error that lateral deviation was non-existent, and that, therefore, the latter type of wave was responsible, in all the cases examined, for the whole of the errors experienced at night in the closed-coil direction-finding system.

These experiments thus constitute an entirely self-contained proof that at night time there are waves arriving in a downward direction polarised with the electric force in a horizontal plane.

It has been shown that good reasons exist for the hope that this system may be successfully developed into a direction-finding system free from night errors and "aeroplane" effect.

These investigations were carried out for the Radio Research Board of the Department of Scientific and Industrial Research. The authors wish to thank Mr. R. NAISMITH for his assistance in the later portions of the experiments described in the paper.

APPENDIX.

SOME EARLY OBSERVATIONS ON AIRCRAFT WITH THE FOUR-AERIAL DIRECTION-FINDER.

BY F. ADCOCK, M.B.E., B.Sc., ASSISTANT IN THE METALLURGY DEPARTMENT.

The authors suggested that as all their observations had been obtained on land transmitting stations, it would be of interest for me to add an account of some experiments which were made with direction-finders on transmissions from aeroplanes in 1918. It was known at that time that bearings or directions obtained by means of the ordinary "closed loop" direction-finders were often quite inaccurate or indeterminable. Experiments with a Hertzian oscillator and with a small spark transmitting set fitted with two equal and symmetrically disposed aerials instead of the more usual aerial and earth arrangements, confirmed the view that the vagaries of the ordinary direction-finders were due to the presence of a horizontal component in the electric force of the radiation emitted by an aeroplane in flight. Further, it was thought that this horizontal component of the electric force of the radiation owed its existence to the two following independent causes :—

(1) An appreciable angle of incidence of the radiation in respect to the ground would cause the wave-front to lean forward.

(2) Owing to the drag of the aerial of an aeroplane in flight, the axis of the "equivalent Hertzian oscillator" of the machine would depart considerably from the vertical. Under these conditions the horizontal electric component from this cause would be non-existent when the aeroplane was flying directly towards or away from the

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examination point, but would attain its maximum value when the machine was flying tangentially with respect to the observer.

The tests mentioned below were made with the aid of an aeroplane flying at a moderate altitude and at distances of 3 to 8 miles from the direction-finding installations, in which circumstances it was believed that any errors of direction would be associated almost entirely with the direction of flight of the machine. Two direction-finding installations were used for taking simultaneous observations on the aeroplane, one equipped with a revolving loop aerial, the other with the fixed experimental aerials devoid, as far as possible, of effective horizontal members. The two installations were erected close to one another but not so near as to cause mutual interference. It was then arranged that the aeroplane fitted with a wireless transmitter should fly over a village in a direction at right angles to that of an imaginary line joining the village in question and the direction-finders. A return journey was then made over the same village, the aeroplane flying in the opposite direction. Wireless transmission took place when the machine was directly over the village, and the two positions of the aeroplane with respect to the direction-finders are shown in Fig. 10. The whole process was repeated over four villages situated at gradually increasing distances from the direction-finders, and the errors in the wireless bearings obtained in each case by both installations are indicated in the table below.

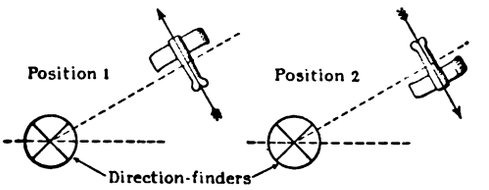

FIG. 10.

TABLE.

Aeroplane flying over.	Approximate distance of aeroplane from both direction-finders.	Approximate true bearing of aeroplane from both direction-finders.	Error of bearing on aeroplane as recorded by direction-finder with revolving-loop aerial.		Error of bearing on aeroplane as recorded by direction-finder with four vertical aerials.	
			Aeroplane in position No. 1 (Fig. 10).	Aeroplane in position No. 2 (Fig. 10).	Aeroplane in position No. 1 (Fig. 10).	Aeroplane in position No. 2 (Fig. 10).
	miles					
Village A ...	3	65°	—	−21°	—	+1°
Village B ...	4	67°	+24°	−33°	−1°	−6°
Village C ...	5½	66°	+31°	−23°	−3°	−5°
Village D ...	8	63°	+40°	—	−2°	+1°

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The results were meagre, but it was clear that large and consistent errors were present in the bearings obtained with the revolving-loop direction-finder. On the other hand the apparatus fitted with the four vertical aerials gave errors in the bearings of only about one-fifth this magnitude. It was also of interest that the errors in the bearings taken with the revolving-loop direction-finder were reversed when the direction of flight of the transmitting aeroplane was reversed, and that these errors showed little tendency to diminish as the distance between the aeroplane and the direction-finder was increased.

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XIX.

THE ATTENUATION OF WIRELESS WAVES DUE TO THE RESISTANCE OF THE EARTH

BY

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ABSTRACT.

The paper calls attention to the present condition of knowledge on the subject of the attenuation of wireless waves travelling over the earth's surface, due to energy absorption by the earth itself. The theories put forward on this problem by SOMMERFELD and ZENNECK are briefly outlined, and it is shown that the latter has taken account only of a special case of the more general theory of SOMMERFELD. The results deduced from this theory have been worked out for some typical practical cases of both short- and long-wave transmission, but, owing to the complete lack of experimental evidence, no practical test of the theory has yet been made. In view of the importance of a knowledge of ground absorption in connexion with the complete study of the propagation of wireless waves of all lengths over the earth's surface, it is highly desirable that a systematic experimental investigation should be carried out in the near future.

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THE ATTENUATION OF WIRELESS WAVES DUE TO THE RESISTANCE OF THE EARTH.

By R. L. SMITH-ROSE, D.Sc., PhD., A.M.I.E.E., Senior Assistant in the Electricity Department, and R. H. BARFIELD, M.Sc., of the Radio Research Station, Department of Scientific and Industrial Research.

[Reprinted from the 'Journal of the Institution of Electrical Engineers,' Vol. 64, 1926.]

I.—*Scope of the Paper.*

During recent years the problem of the propagation of wireless waves over the earth's surface has been attacked by several experimenters by the systematic measurement of the intensity and direction of the fields arriving at a receiver from a distant transmitting station. The results of this research show generally that two series of waves arrive at the receiving station from the transmitter. The first of these is the "direct" wave which is propagated along the surface of the earth following its curvature by diffraction. The second series comprises those "indirect" waves which have travelled by way of the upper regions of the atmosphere, having reached the receiver by one or more suitable deflections. In a complete study of the problem it is evidently very desirable to ascertain what are the relative contributions of the direct and indirect waves to the resultant field strength under all practical conditions of wireless transmission over the earth's surface; and, further, to account on satisfactory theoretical grounds for all the experimental results obtained.

The complete problem may be conveniently divided into two parts: first, that of determining the field strength at distances from the transmitter so great that the curvature of the earth has to be taken into account; and second, that of determining this quantity at points relatively close to the receiver, *i.e.*, over distances for which the earth may be considered flat. The first part of the problem has in the past proved to be of greater interest and has accordingly had a great deal of attention paid to it. This has probably been due to the fact that, within the range of wave-lengths to which commercial communication has been restricted in the past, no difficulty was experienced in obtaining satisfactory transmission over distances which come within the second category. Within this region it is evident that the direct wave will appreciably contribute to the resultant field strength, and it is thus important to have an exact knowledge of the extent to which this wave is attenuated due to the finite conductivity of the earth. This question is becoming more important with the use of shorter wave-lengths, and cases are already on record in which the direct wave has become too weak for detection

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or measurement at distances much less than those at which the indirect wave had become the sole contributor to the field at the earth's surface. For example, in some observations published by A. H. TAYLOR,* it has been shown that for wave-lengths of 40 metres and less, and with a given power, the maximum range of transmission directly along the earth's surface was less than 100 miles, and that no signals could be received at greater ranges until a distance exceeding 400 miles was reached. At these greater distances the signals were evidently due to the reception of waves from the upper atmosphere. The intervening range in which neither the direct nor the downcoming waves can be received is commonly referred to as the "jump over" or "skipped distance."

This attenuation problem has been studied theoretically by SOMMERFELD† and ZENNECK,‡ but as far as the authors are aware, the formulæ which these investigators obtained have never been experimentally tested. In SOMMERFELD'S work, results were obtained which can be employed for the calculation of attenuation over a very large range of wave-lengths, including the shorter waves which are now coming into practical use and on which the matter has assumed greatly increased importance. Owing to the lack of experimental data upon the subject, an investigation has been planned, and will be carried out as soon as opportunity permits. Pending the results of this, it has been considered useful to give a brief outline of SOMMERFELD'S work, together with a series of attenuation curves calculated for such cases as are of interest.

II.—*Outline of Sommerfeld's Theory.*

Given a Hertzian oscillator situated on the earth's surface, SOMMERFELD obtains an expression for the intensity of the field at any other point on the earth, taking into account the conductivity and dielectric constant of the ground. Throughout his work he neglects the effect of the curvature of the earth and assumes the atmosphere to be homogeneous. (It may be pointed out here that DUDDELL and TAYLOR'S experimental results for transmission over the sea, as mentioned on p. 404, show that the effect of the curvature of the earth up to a distance of 60 miles is practically negligible.) The formulæ obtained by SOMMERFELD, which are most suitable for numerical calculation, are simplified by assuming $2\sigma/f$ to be much greater than unity, where σ is the conductivity of the earth in electrostatic units, and f is the frequency corresponding to the waves. Taking the accepted value of σ as 10^{8} § for land in this country, then $2\sigma/f$ will exceed 10 for all wave-lengths above 15 metres, and so this assumption is applicable to almost the entire range of practical cases.

* A. H. TAYLOR, 'Proc. Inst. Radio Eng.,' Vol. 13, p. 677, 1925; also TAYLOR and HULBERT, 'Q.S.T.' Vol. 9, p. 12, 1925.

† A. SOMMERFELD, 'Ann. der Phys.,' Vol. 28, p. 665, 1909.

‡ J. ZENNECK, *ibid.*, Vol. 23, p. 846, 1907.

§ This corresponds to approximately 10^{-13} in electromagnetic units, or to 10,000 ohms per cm. cube.

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The notation employed for this résumé is as follows : Principal axes are assumed with the transmitter at the origin and Ox the direction of transmission.

K = dielectric constant of the earth,

σ = conductivity of the earth,

f = frequency,

$j = \sqrt{(-1)}$,

$K' = K - (j2\sigma/f)$,

$\mu = \sqrt{K'}$ = " refractive index " of the earth,

P = vector potential = distance rate of change of the radiated magnetic field $= \partial H/\partial x$.

$\rho = \frac{\pi}{\mu^2} \cdot \frac{x}{\lambda}$ = " numerical distance."

Beginning with the classical equations of CLERK MAXWELL and applying boundary conditions, SOMMERFELD obtains a series for P which is finally reduced to forms suitable for calculation. The approximate solutions differ according to whether the numerical distance ρ is large or small.

(*a*) If ρ is large, say not less than 10, then the following equation applies :—

$$P = -\left\{\frac{1}{2\rho} + \frac{1 \cdot 3}{2 \cdot 2\rho^2} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2\rho^3} + \ldots\right\} \frac{e^{j2\pi x/\lambda}}{x}. \quad \ldots\ldots \quad (1)$$

When ρ is very large this reduces to

$$P = \frac{A\lambda^2}{x^2}, \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

i.e., for a given current in the oscillator the radiated field intensity is proportional to the square of the wave-length and inversely proportional to the square of the distance.

(*b*) If ρ is small, the approximate formula is as follows :—

$$P = (u - jv) \frac{e^{j2\pi x/\lambda}}{x}, \quad \ldots\ldots\ldots\ldots \quad (3)$$

where

$$u = 1 - \frac{2}{1}\rho + \frac{2 \cdot 2}{1 \cdot 3}\rho^2 - \frac{2 \cdot 2 \cdot 2}{1 \cdot 3 \cdot 5}\rho^3 + \ldots$$

and

$$v = \sqrt{(\pi\rho)}\, e^{-\rho}.$$

From formulæ (1) to (3) it is possible to calculate the intensity of the radiation field at any distance over which it can be assumed that the curvature of the earth produces no appreciable effect.

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III.—*Discussion of Calculations from Sommerfeld's Equations.*

In Figs. 1 to 4 the values of the radiated field have been calculated for distances up to 150 km. and for wave-lengths from 30 to 1,000 m. Graphs are given for transmission over sea water (assuming $\sigma = 10^{10}$ E.S.U.*) and over land for which $\sigma = 10^{8}$ E.S.U.†

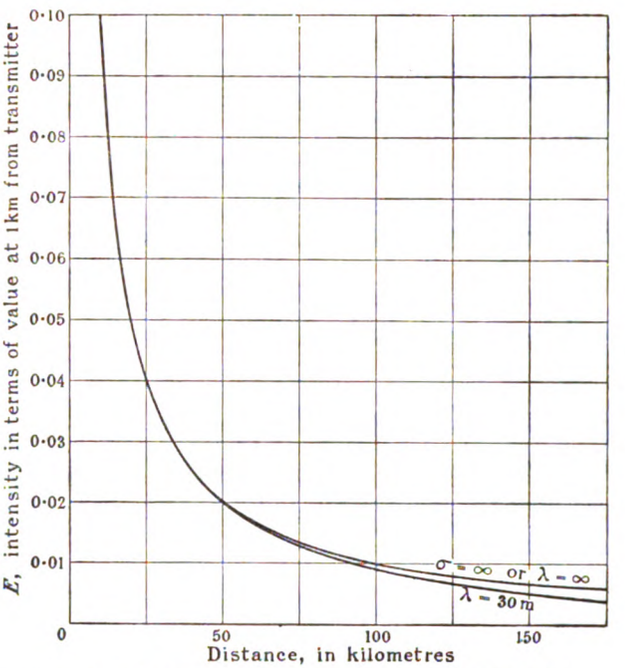

FIG. 1. Attenuation over Sea ($\sigma = 10^{10}$ E.S.U.).

In Figs. 1 and 2 the actual field intensity at various distances is shown in terms of the value at 1 km. from the transmitter. In Figs. 3 and 4 the results are shown as the ratio of the intensity of the field which pertains for transmission over sea or land to the intensity which would be obtained if the earth were a perfect conductor.

In each of Figs. 1 to 4 a curve is given as applying to the condition $\lambda = \infty$ or $\sigma = \infty$. It is of course realised that in considering the radiated field intensity, the minimum distance of the point under consideration from the transmitter is of the order of a wave-length. The above hypothetical curves are, however, inserted as showing the upper limits of the field intensity which could be obtained either by using very long wireless waves or by transmission over a perfectly conducting earth.

(*a*) *Transmission over Sea.*—Figs. 1 and 3 demonstrate very clearly that for wave-lengths above 100 m. the attenuation over sea is practically identical with that which would hold for the ideal case of a perfect conductor, the field intensity decreasing inversely as the distance. With a wave-length of 30 m., however, a noticeable departure from the ideal case is observed, and at a distance of 150 km. the field has fallen to about 0·8 of its ideal value.

* Equals approximately 10^{-11} in E.M.U., or 100 ohms per cm. cube.

† Equals approximately 10^{-13} in E.M.U., or 10,000 ohms per cm. cube.

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(*b*) *Transmission over Land.*—When the transmission is entirely over land the effect of the energy absorption due to the higher resistance is very marked on wave-lengths below 1,000 m., as shown in Figs. 2 and 4. The 1,000-m. wave is attenuated over land to approximately the same extent as the 30-m. wave over sea. On a wave-length of

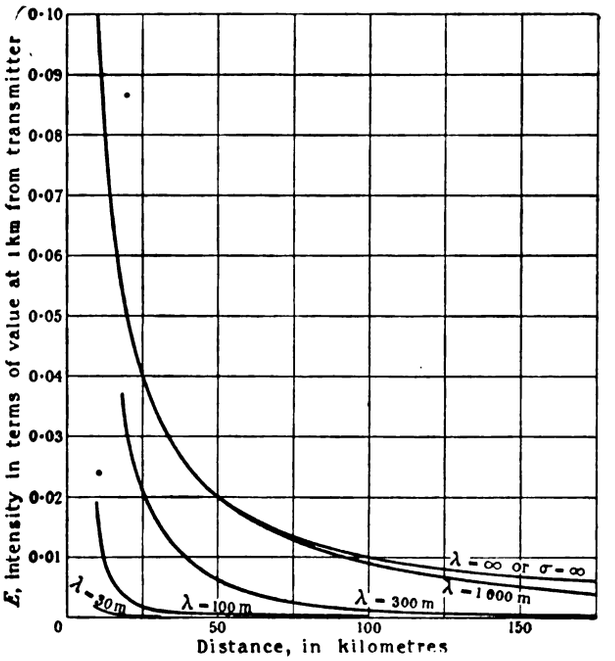

FIG. 2. Attenuation over Land ($\sigma = 10^8$ E.S.U.).

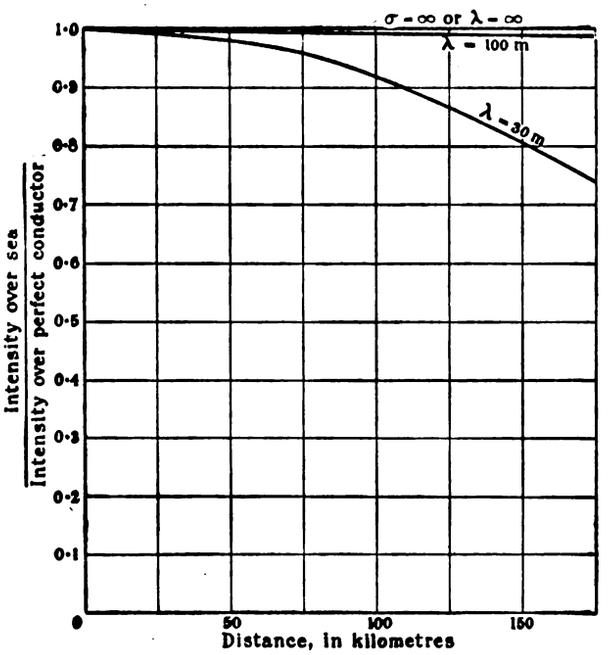

FIG. 3. Attenuation over Sea ($\sigma = 10^{10}$ E.S.U.).

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300 m., however, the intensity at a distance of 150 km. over land is only 0·08 of the ideal value or of its value for transmission over sea to the same distance. On the shorter wave-lengths, the attenuation becomes still more marked. For example, on a wave-length of 30 m. the intensity at only 10 km. over land is reduced to 0·01 of its ideal or approximate over-sea value, which is equivalent to 0·001 of the intensity at 1 km. from the transmitter.

Such examples serve to illustrate very clearly the great disadvantage of the shorter wave-lengths for transmission over small and medium distances, at which the curvature of the earth and the effects of the upper atmosphere are negligible.

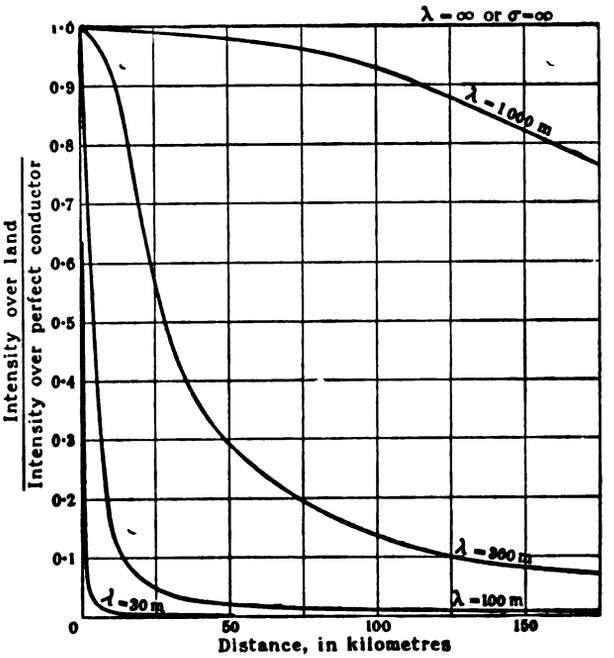

FIG. 4. Attenuation over Land ($\sigma = 10^8$ E.S.U.).

IV.—*Zenneck's Analysis as a Particular Case of Sommerfeld's Theory.*

In his analysis of the effect of the finite conductivity of the earth on wireless-wave propagation, ZENNECK simplified the problem by making the assumption that at a large distance from the transmitter the wave may be considered plane, thereby neglecting the ordinary attenuation of the wave due to spreading. On this basis ZENNECK showed that the decrease in intensity of the wave could be represented by an expression of the form

$$E = Be^{-\alpha x}, \qquad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

where α is an attenuation constant depending upon the constants of the earth and the wave-lengths, viz., $\alpha = \pi/(\mu^2\lambda)$. It is thus evident that the quantity αx is identical with SOMMERFELD'S ρ, and that this part of the attenuation is embodied in the value of

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v which forms part of equation (3) above. Under certain conditions when u approaches zero, equation (3) for the intensity takes the form of equation (4). Indeed, SOMMERFELD pointed out this in his theoretical work and showed that this constituent part, which may be called " ZENNECK's wave," is of the nature of a surface wave, and that, apart from its attenuation due to earth resistance, the field intensity varies inversely as the square root of the distance.

It is clear, therefore, that ZENNECK provided a partial solution which is valid only under particular conditions, while SOMMERFELD'S solution is quite general. Since SOMMERFELD includes ZENNECK's particular case, there would appear to be no point in making any calculations from the latter's formulæ. It is interesting, however, to ascertain under what conditions ZENNECK's theory gives an approximate solution. This information is readily obtainable from SOMMERFELD'S theory and is recorded in Table I.

TABLE I.

Wave-length.	Range of transmission over which equation (4) is valid.	
	Over land ($\sigma = 10^8$).	Over sea ($\sigma = 10^{10}$).
m.	km.	km.
10,000	10,000 to 32,000	Too large for practical use
3,000	900 to 3,100	Too large for practical use
1,000	100 to 320	Too large for practical use
300	9 to 31	9,800 to 29,000
100	1·0 to 3·1	1,100 to 3,300
30	0·09 to 0·3	98 to 290

This table shows that the applicability of ZENNECK's solution is very limited. Thus, on land and with a wave-length of 1,000 m. and upwards, it is never valid in the first 100 km.; and at distances much larger than this the curvature of the earth would have to be taken into account, and both ZENNECK's and SOMMERFELD's formulæ become invalid. For wave-lengths less than 1,000 m. the range of validity over land is very small. Thus on 300 m. it is only 22 km., and on 100 m. it is only 2 km. For over-sea transmission the distances over which ZENNECK's formula could be used are not small enough for the earth to be assumed plane for any wave-lengths above 30 m. It appears, therefore, that on nearly all wave-lengths at present in use for commercial purposes, the conditions under which ZENNECK's theory holds good are so limited as to render his formulæ of little value for practical purposes.

V.—*Previous Experimental Results.*

As far as the authors have been able to ascertain, the only previously published papers which provide any reliable data upon the problem under discussion are those of DUDDELL

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and TAYLOR, describing experiments made as long ago as 1905,* and of BOWN and GILLETT published in 1924.† It is significant of the lack of interest in this subject that apparently no investigation of it was carried out in the intervening 19 years.

The first of the above papers describes two sets of experiments, the first made over land in Bushy Park, Teddington, and the second over sea across the St. George's Channel. In both experiments a spark transmitter was used, and was moved to various positions, while the corresponding current in the receiving aerial was measured by means of a thermo-galvanometer. In the first series of experiments carried out in Bushy Park the range of transmission was quite small, never being greater than 7,500 feet (2·25 km.). Also the site of the experiments was far from uniform, the open grass spaces being interspersed among clumps of large trees of several hundred yards in extent. In view of these considerations and of the fact that the wave-length employed was 400 to 500 feet (120 to 150 m.), and of the probable serious effect of the trees on such short waves, it is not considered that the conditions of the experiments were such as to provide data sufficiently reliable for comparison with any theoretical formulæ. The experiments did, however, show quite clearly that the intensity of the waves decreased much more rapidly than the inverse distance from the transmitter, a fact which was probably due to absorption of energy by the ground and trees.

The second series of experiments of DUDDELL and TAYLOR were much more satisfactory, being carried out over an open-sea path at distances up to 60 miles (96 km.). The wave-length employed is not stated, but was probably of the same order as above. Under these conditions the results of the measurements showed that the received current and, thus, the field intensity of the waves, were practically inversely proportional to the distance, as would be the case for uniform spreading over a plane perfect conductor. The product of received current and distance did show a steady but small decrease over the range of transmission. This decrease represents the sum of that corresponding to the small attenuation of such waves when transmitted over sea-water, as indicated from the theoretical curve in Fig. 3, and the decrease which would be caused by the slight but appreciable curvature of the earth over a distance of 100 km.

In BOWN and GILLETT'S experiments it is perhaps natural to find that the conditions were a little more satisfactory. The measurements were made with a portable coil receiver on the undamped-wave transmissions from broadcasting stations in New York and Washington, U.S.A. In the paper referrred to, the results are well represented both in the form of curves showing the decrease of field strength with distance, and as contour maps giving the curve of uniform field distribution round the transmitting station. On the wave-length of 469 m. employed, these results show very clearly that the attenuation is small over river water, and increasingly greater for moist grass-land and dry sandy soil. The experiments also clearly demonstrate the high absorption which is experienced when the transmission is over tall iron-frame buildings, as in the New York City area.

* W. DUDDELL and J. E. TAYLOR, 'J. Inst. Elec. Eng.,' Vol. 35, p. 321, 1905.

† R. BOWN and G. D. GILLETT, 'Proc. Inst. Radio Eng.,' Vol. 12, p. 395, 1924.

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By dissecting the curves for the different natures of the path of transmission, BOWN and GILLETT compared the measured results with the well-known Austin-Cohen transmission formula, which is as follows :—

$$\text{Field strength} = \left\{377\,\frac{h_s I_s}{\lambda x}\right\} e^{-\alpha x/\sqrt{\lambda}} \text{ microvolts per metre,} \quad \ldots\ldots \quad (5)$$

where

h_s = effective height of the transmitting aerial in metres,
I_s = transmitting aerial current in amperes,
λ = wave-length in km.,
x = distance in km.

The attenuation due to resistance absorption is expressed by the quantity $e^{-\alpha x/\sqrt{\lambda}}$ and the measured values of α are given in the paper as :—

0·028 for dry sandy soil,
0·009 for moist soil,
0·0025 for water,

while the value of 0·0015 is commonly used in the above formula for over-sea transmission in daylight. As this attenuation formula is of an empirical nature, however, and, further, as it needs verification for other wave-lengths, it is not of much value in making comparisons with the theoretical formulæ discussed in Sections II and III. Furthermore, the fact that the measurements were made over ranges which comprised patches of soil of a varying nature, of which no information was obtained as to the effective conductivity, would also make a direct comparison with theory very difficult.

VI.—*Conclusions.*

In this paper the authors have thus arrived at the conclusion that, for wireless-wave transmission over comparatively short distances, under conditions where it is necessary to take into account the resistance of the earth, SOMMERFELD's analysis alone leads to a means of calculating the attenuation for all practical cases. The graphs in the paper provide in summarised form the practical information to be derived from this theory. Owing to the complete lack of experimental evidence on the subject, it is as yet impossible to state whether the theoretical values are in accordance with those obtained in practice.

It is evident that what is wanted is a series of careful measurements giving either the absolute or the comparative field strengths at different distances up to say 50 or 100 km., the transmission being made over ground as uniform as possible, the conductivity of this ground having been measured by an independent method such as has already been employed in England.* It will, of course, be a condition of these measurements that any waves received from the upper atmosphere have no perceptible effect.

* SMITH-ROSE and BARFIELD, 'N.P.L. Collected Researches,' Vol. 21, Paper 15, p. 317 (or 'Proc. Roy. Soc.,' A, Vol. 107, p. 587, 1925.

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XX.

THE PROPAGATION OF RADIO WAVES

BY

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SOMETIME ASSISTANT IN THE ELECTRICITY DEPARTMENT.

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ABSTRACT.

The paper gives the results obtained from a systematic study of the received intensities of various long-wave transmitting stations at four receiving stations over a period of nearly two years.

The weekly averages of the results are studied, and it is shown that they are inconsistent with any law of steady decay. The results also show very marked periodic variations.

It is suggested that all the observed effects can most easily be explained on the assumption of an upper refracting layer. This is supported by the results obtained from plotting an intensity-distance curve in detail by means of a portable set, and by the study of the systematic changes which take place daily during the sunset period.

Various conclusions are drawn from these results, but it is pointed out that considerable further experimental investigation of many of the observed phenomena is necessary.

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THE PROPAGATION OF RADIO WAVES.

By J. HOLLINGWORTH, M.A., D.Sc., M.I.E.E., Sometime Assistant in the Electricity Department.

[Reprinted from the 'Journal of the Institution of Electrical Engineers,' Vol. 64, 1926.]

Introduction.

The study of the propagation of radio waves through the space between the transmitting and receiving aerials is one that has attracted many workers, as it is of fundamental importance. Ever since the early days the discrepancies between actual and expected results have demanded explanation. The problem has often been attacked from the theoretical side; but in those days, largely owing to the limitations of experimental technique, the underlying physical data were indefinite, and consequently very varying results were obtained. With the development of the thermionic valve and the resulting more accurate measuring apparatus, however, it has become possible to put the experimental side on a firmer basis, and so provide more definite information for the theoretical solution.

Even from the experimental side there are two ways of approaching the subject, though the line of demarcation is not always sharply defined.

The first is primarily concerned with obtaining data for reception and transmission purposes, and so is most concerned with long-distance transmission under working conditions. Valuable theoretical results may and do follow from this, but they still remain a secondary factor. On the other hand immediate practical results may be subordinated, the measurements being viewed chiefly as a convenient means of examining the electrical state of the intervening medium, in which case it often happens that the actual figures obtained have no direct practical application. In the following work the second of these two alternatives has always been given most consideration, and consequently, after the first preliminary trials, all experiments have been chosen with a view to illustrating some particular feature to the best possible advantage.

Apparatus.

The apparatus is substantially the same as that described in a previous paper,* the only considerable alteration being the substitution of a Dye transformer† for the calibrated mutual inductance. Actual experience has, of course, resulted in a certain

* J. Hollingworth, 'N.P.L. Collected Researches,' Vol. 18, Paper 8, p. 141 (or 'J. Inst. Elec. Eng.,' Vol. 61, p. 501, 1923.

† D. W. Dye, 'N.P.L. Collected Researches,' Vol. 22, Paper 8, p. 133 (or 'J. Inst. Elec. Eng.,' Vol. 63, p. 597, 1925.

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number of detail modifications both in design and in operation, the chief one being the addition of a string galvanometer so that readings can be taken on the ordinary routine of a station; but the fundamental principle remains unchanged.

General Principles.

This series of experiments differs from the majority of those which have already been made on this subject in several important particulars. The chief of these are:—

(*a*) Distances are comparatively short.

(*b*) Simultaneous observations have been taken at different stations whenever possible.

(*c*) Only daylight transmissions have been considered.

As these factors are not accidental, but part of a settled policy, they require a short consideration.

Experiments of this nature differ from many others in containing an excessive number of unknown and uncontrollable factors. In the first case, many of the causes of variations are at present altogether unknown, and the experimenter has practically no power of controlling them, or even of being able to repeat an experiment under reasonably similar conditions. Secondly, although on ideal grounds the transmitting station should be under his control or, as in the case of the special "URSI" signals, be working under carefully observed conditions, it is in practice impossible to extend this to all the observations which become necessary; so that, even in the case of the known possible causes of variation, the control is still incomplete.

Scientifically, this means that the number of independent variables is unusually large, so that unless the routine of observation is controlled as rigidly as possible, masses of results are obtained under such indeterminate conditions that, except on a purely statistical basis, no attempt can be made to sort out the various causes. In the early days of the work this rule was, of course, less rigidly applied, but as soon as an inkling into possible causes had been obtained, subsequent experiments were planned to illustrate that particular point under the most favourable conditions.

It is not denied that the results of long-distance work are in practice the most important and yield the most surprising results. They are also the ones to which most attention has been given by other experimenters, so that a considerable amount of information is already available about them. If, however, the matter be considered in the same light as any other physical experiment, the natural method is to take the simplest cases first, and to work up from them to the more complex. After all, the mechanism of propagation must always be the same, though it may exhibit its effects differently at different distances.

Comparatively short-distance work appears to offer an opportunity for reducing the uncertainty with regard to some of the independent variables, as it seems reasonable to suppose that there is a chance of natural phenomena being more constant over small areas than over large ones. In particular, it removes the special cause of difficulty in

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long-distance transmission, in which the path contains both day and night sections. It also tends to higher accuracy of measurement, owing to the more powerful signals in relation to disturbances.

It appears very doubtful whether any satisfactory generalisations can be drawn from measurements of a single transmitting station at a single receiving station. Such a process assumes that the situation of the receiver is ideal and unaffected by local conditions, and also that the law of propagation is perfectly regular and uniform, and it offers no verification of either of these facts. Consequently, in these experiments four receiving stations have been used, at Slough, Manchester, Glasgow and Aberdeen. These sites had to be chosen more with a view to the facilities available than on scientific grounds, but the results obtained seem to justify their choice.

Experimental Results.

The basis of all the earlier work has been the " URSI " signals. These are special transmissions arranged under the auspices of the Union Radio Scientifique Internationale, and consist of a preliminary call—URSI de . . . . —then a statement of the exact wave-length and aerial current of the previous day's transmission, followed by a steady dash lasting two minutes. Those of which most use has been made are :—

(i) From GBL (Leafield) daily except Sundays and public holidays at 1400 G.M.T. ; $\lambda = 12{,}350$ m. ; $I = 200\,\alpha$ (nominal).

(ii) From UA (Nantes) daily at 1415 G.M.T. ; $\lambda = 9{,}000$ m. ; $I = 180\,\alpha$.

(iii) From LCM (Stavanger) daily at 1000 G.M.T. ; $\lambda = 12{,}140$ m. ; $I = 200\,\alpha$. (In this case the signal consists of a series of 20-second dashes.)

A signal is also sent from LY (Bordeaux) at 1955 G.M.T. ; $\lambda = 18{,}900$ m. ; $I = 480\,\alpha$; but this has not been much used up to the present, owing to its proximity to the sunset period.

Routine observations have been kept on these stations since June or July, 1924, and, in the case of Nantes, at Manchester since March, 1924. Slight interruptions have been caused at times owing to the fact that to obtain trustworthy results it is vital to employ competent experimenters, who, in general, could only give part of their time to the work.

These results were sent to the Laboratory at Teddington, where they were reduced to a fixed aerial current and each set was averaged weekly. The system of weekly averages, each involving, as it does, from two to six readings only, was adopted only after some consideration, as it was felt that they were rather small groups to average.

The plotting of daily results was soon found to be too cumbrous, whereas monthly averages would have enormously lengthened the work, and might possibly have hidden

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interesting transient results of shorter period. On the whole, the study of the curves seems to show that the system of weekly averages has been fairly justified.

The results require analysis on both a time and a space basis, *i.e.*, a study of the week-by-week variations, at each receiving station, of each transmitter, and also of its simultaneous distribution at all four receiving stations. In view of the number of graphs involved, and the widely different scales to which they have to be plotted, it is necessary at times to plot selections from them in various fashions, but for the present all results will be derived from the weekly figures referred to above.

Seasonal Intensity Graphs.

Fig. 1 gives the weekly averages of the observations on Nantes and Leafield at the four receiving stations over the period June, 1924–December, 1925.

Considering first the early months, the most striking feature is that Manchester recorded, on Nantes, signals equal to and even greater than those at Slough, although the distances are 695 km. and 485 km. respectively. It was also noticed that, on the few individual days when variations did occur, they usually appeared at all stations, but not necessarily in the same direction. An unusually high value at one station was generally accompanied by a low value elsewhere.

During September, 1924, although the change in weekly averages was mostly small, day-to-day variations became much larger; and at the same time it was noticed at Slough, where simultaneous readings of direction were taken on another set, that the bearings were becoming unsteady and the minima poor. In October, however, big changes began to occur, culminating in the sudden and enormous variations in the week 25th October–1st November. Consideration of the daily logs shows that actually the majority of the changes occurred between the 27th and 30th October, and, as it was really this which gave the first suggestion of a possible cause, it requires discussion in detail.

The first and most striking characteristic was the simultaneity of the changes. In these few days signals from stations as far apart as Nantes and Stavanger, at receivers as far apart as Aberdeen and Meudon, showed abrupt changes, amounting in some cases to over 100 per cent., and the figures published by Prof. MESNY in ' L'Onde Electrique ' of readings on Bordeaux at Meudon show the same variation (see Fig. 2), even though at this time of year this signal is sent several hours after dark. Traces of the effect are also visible in Dr. AUSTIN's figures for observations on some of the European stations at Washington,* and in a recent paper by Captain ROUND and Mr. T. L. ECKERSLEY† Fig. 14 seems to indicate a similar phenomenon in previous years, though the exact time of its occurrence is not so definitely marked. The effects were not merely transient, as in most cases, although the day-to-day unsteadiness was very marked; there was in the following weeks no definite reversion to the summer values.

* AUSTIN, ' J. Wash. Acad. Sci.,' Vol. 15, p. 227, 1925.

† ROUND and ECKERSLEY, ' J. Inst. Elec. Eng.,' Vol. 63, p. 943, 1925.

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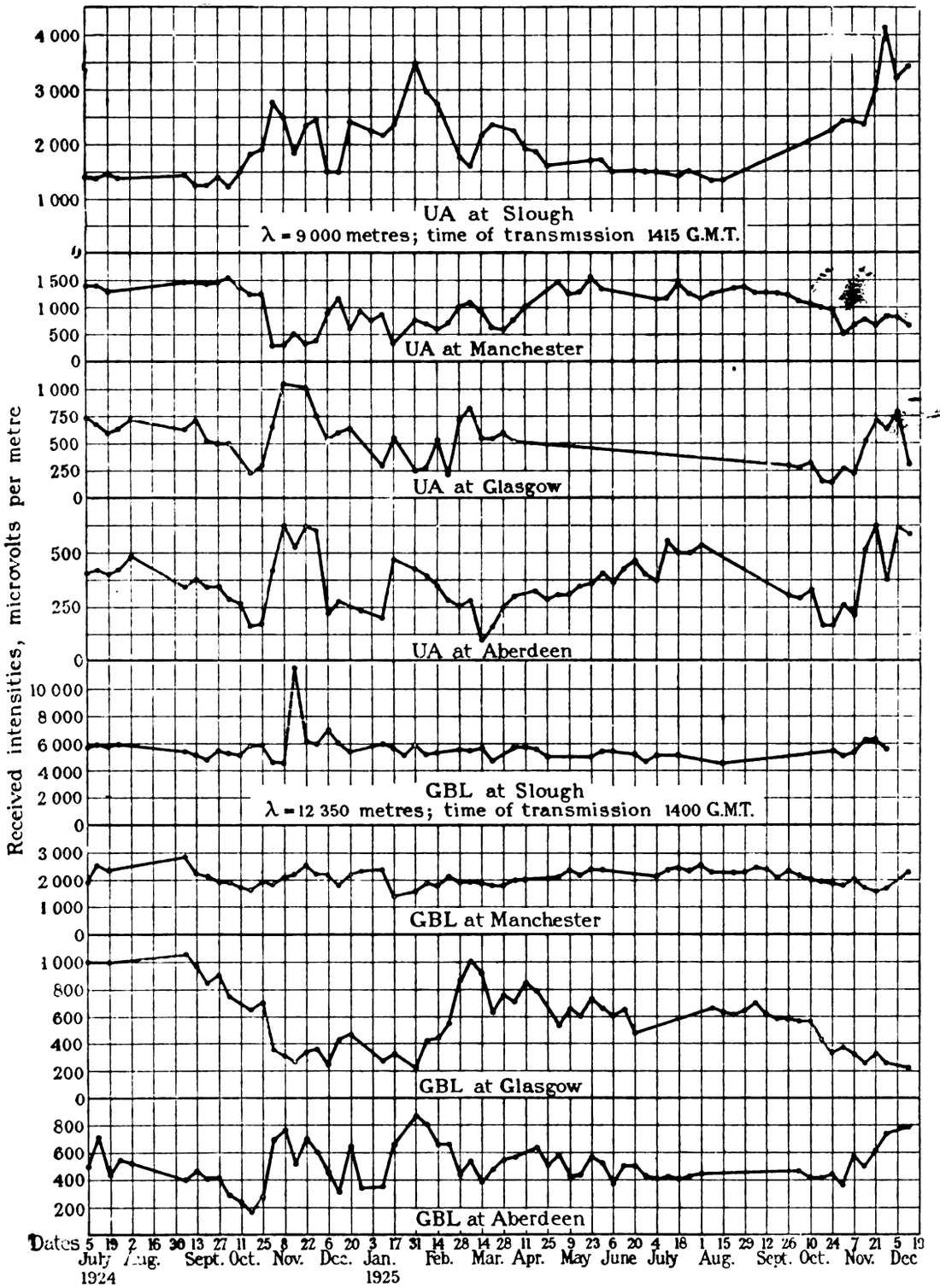
Received intensities, microvolts per metre
UA at Slough
λ = 9 000 metres; time of transmission 1415 G.M.T.
UA at Manchester
UA at Glasgow
UA at Aberdeen
GBL at Slough
λ = 12 350 metres; time of transmission 1400 G.M.T.
GBL at Manchester
GBL at Glasgow
GBL at Aberdeen
Dates
July Aug. Sept. Oct. Nov. Dec. Jan. Feb. Mar. Apr. May June July Aug. Sept. Oct. Nov. Dec
1924
1925

FIG. 1. Diagram showing Weekly Average Intensities of "URSI" Signals from UA (Nantes) and GBL (Leafield) at the Four Fixed Receiving Stations.

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In 1925 the results were very closely similar, in that, during the spring, signals gradually steadied down again to practically the same values as those during the previous summer. Iu the autumn the same feature of the abrupt October change again occurred on practically the same days. Another fact, perhaps equally striking, is that after this date the day-to-day variations were enormously greater than before it.

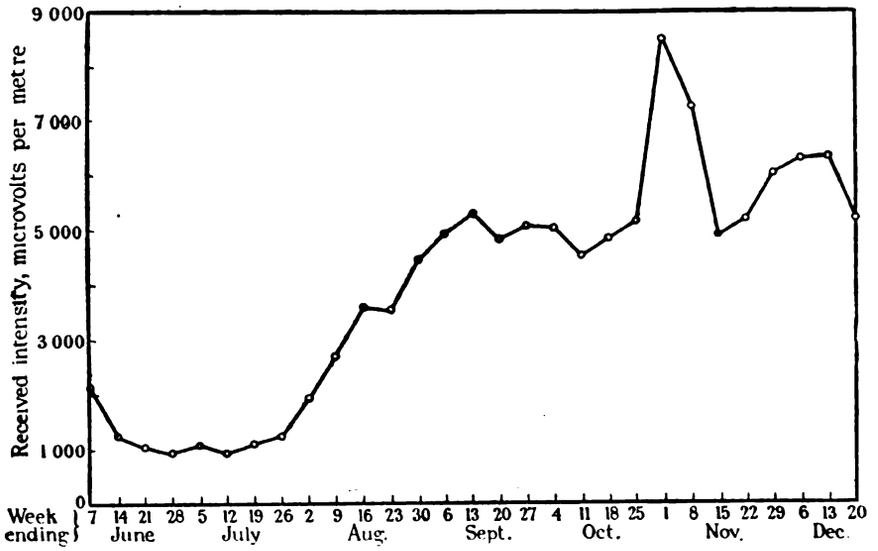

FIG. 2. Weekly Average of " URSI " Signal from Bordeaux (LY) received at Meudon by Prof. MESNY. $\lambda = 1{,}920$ m.; time 1955 G.M.T.; distance 520 km.

For instance, in the week ending November 29, 1925, the daily readings of Nantes at Manchester were 1,320, 1,390, 50, 209, 550; and at Aberdeen 560, 500, 241,115, so that with results like this a weekly average figure has not much meaning. This will be referred to later.

It is hardly possible to avoid the conclusion that these changes were not due to local variations, but to some main cause operative over a large area, and the possibility is also suggested that all signal-strength variations might be due to this cause, differing in degree rather than in kind. Also, in view of the fact that the signals from Leafield at Glasgow and from Stavanger at Aberdeen (see Fig. 3) were practically equal, although the former path is entirely over land and the latter entirely over sea, it does not seem possible to admit the conductivity of the ground as a major controlling factor. It seems most unlikely that this conductivity should remain constant for over four months, and then change suddenly and completely in two days, especially as it will be remembered that 1924 was an extremely wet year, and there are no signs of permanent weather change about this date. Also, SMITH-ROSE and BARFIELD have shown* that in the

* SMITH-ROSE and BARFIELD, 'N.P.L. Collected Researches,' Vol. 21, Paper 15, p. 317 (or 'Proc. Roy. Soc.,' Vol. 107, p. 587, 1925).

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British Isles, at any rate, the conductivity of the soil for long wave-lengths is of a high order.

Further, it will be noticed that at times during the 1924 winter period the intensity

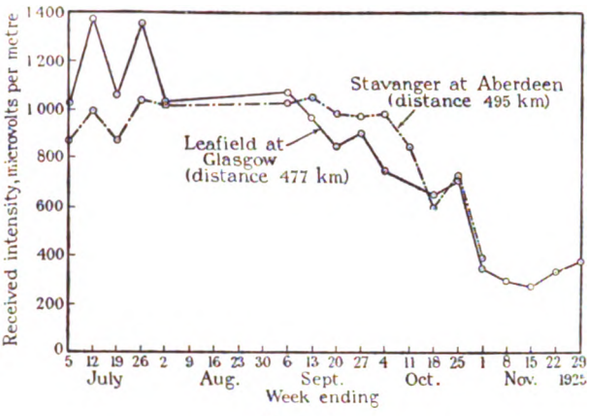

FIG. 3.

of the signals from Nantes at Manchester is actually less than that at Glasgow and Aberdeen, whereas in the case of Leafield the Aberdeen figures are above the Glasgow figures. The various distances are given in Table I.

TABLE I.—Distances between Transmitters and Receivers.

	Slough.	Manchester.	Glasgow.	Aberdeen.
	km.	km.	km.	km.
Nantes (UA)	485	695	973	1,100
Leafield (GBL)	78	184	477	590
Stavanger (LCM)	911	775	680	495
Northolt (GKC)	13	243	540	630
St. Assise (FT)	396	635	930	1,000

For medium distances, therefore, it seems that there is no justification for holding to any law of steady decay except for purely practical average purposes, because to explain the above figures on such a basis, modified by purely local conditions at each station, would involve a large number of simultaneous and elaborate but totally independent coincidences.

It remains to be seen if all these effects can be approximately accounted for by a single cause.

Intensity-Distance Distribution.

To discuss the intensity-distance distribution it is necessary to make use of space-distribution curves, *i.e.*, to plot intensity against distance and not against time. Such a

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curve is given in Fig. 4, but the ordinate scale is unlike the previous ones owing to the fact that, due to the ordinary attenuation, the actual intensities at the various stations are widely different. For the moment we are more interested in the relative variations than in the absolute values, consequently the ordinates in Fig. 4 are the ratio of the monthly average intensity at any point to the mean annual intensity at that point. By this means the relative changes at each station can be studied on a common scale. As at the moment it is the change during October which is under consideration, the two

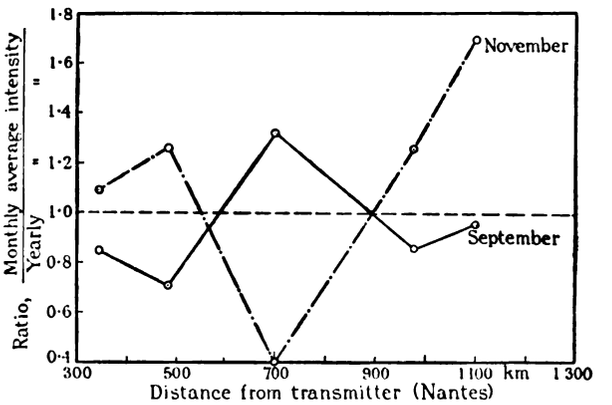

FIG. 4.

curves in Fig. 4 are derived from the monthly averages in September and November. The September curve is closely similar to those of the preceding months and may be taken to represent the average summer distribution; October is deliberately omitted, being treated as a transition period. The characteristic features of these curves are clearly the alternation of high and low values, and also the complete reversal of form between the two months.

Curves of this nature at once suggest something in the nature of an interference phenomenon, so that from this point the primary object of subsequent experiments has been to study this effect under the most favourable conditions.

Now if, as suggested by this and previous results, the intensity-distance distribution is not a smooth curve, it follows immediately that a very much larger number of points on it are required than can possibly be obtained from the four fixed stations; so it was decided to proceed immediately with the construction of a portable measuring set, and meanwhile to carry on with the steady routine of " URSI " signals with a view to their future interpretation.

A study of Fig. 1, and more especially of the daily logs, shows that for an experiment of this nature, which would inevitably occupy some weeks, the work could only be done at a period of the year at which the day-to-day consistency was highest. This limits the possible period to from the middle of May until the middle of September, and consequently the early months of 1925 were occupied in the development and construction of the portable set. The work gained added interest about this time from the publica-

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tion of an article by Prof. APPLETON and Mr. BARNETT,* suggesting the existence of a similar phenomenon on shorter waves; but unfortunately their experimental method was not practicable in the case of long-wave high-power stations. Various delays intervened, and finally it was impossible to start work with the portable set before the middle of June, 1925.

Although the original ideas had been derived from measurements of the signals from Leafield and Nantes, neither of these stations was quite suitable for the purpose of this later experiment. In the case of Nantes, its working hours are short and irregular, and for this sort of work it is vital to have a station working practically continuously, as when on tour it is impossible to guarantee arrival at a fixed spot at a fixed time. The geographical situation of Leafield is also not altogether satisfactory, as an inspection of the figures already obtained suggested that the phenomenon would not in this case become strongly marked until the north of Scotland was reached, which would involve an excessive amount of travelling, together with a possibility of not being able to extend the tour far enough away from the station to obtain the required result. Choice finally fell on St. Assise (FT; $\lambda = 14{,}350$ m.). This station appears to work almost continuously during the most suitable observing hours (10 a.m. to 4.0 p.m.), and it gives a powerful signal, which is most convenient since the limitations involved in making a set portable lead to a slight sacrifice of sensitivity and accuracy. Preliminary tests also showed that the signal usually maintained a very steady value throughout each period of working.

Moreover, reports from the fixed observing stations during May showed that, whilst St. Assise was well received at Glasgow and Aberdeen, considerable difficulty was experienced with it in Manchester, where signals were, on an average, only about one-third of the strength of those at Aberdeen. It was thought that either this was a very marked case of interference, or that Manchester conditions were abnormal, so that a tour on this station might enable a definite decision to be reached.

Five tours were therefore made, four being confined to working as nearly as geographical conditions would permit on the great circle from FT to Aberdeen, and one with the definite object to getting as far away from this great circle as possible, in order to see whether the effects were confined to any particular direction. During these tours control observations were kept at one, at least, of the fixed stations.

In an experiment of this nature, besides the question of day-to-day consistency, there is also the factor of suitable sites, as, especially in hilly country, the choice of what appears to be an ideal position for a site is not always available. Of the actual sites on which the readings were obtained, some did certainly appear to be very good, being situated in open country free from steep hills, heavily wooded country, or long-distance telegraph lines. Such good sites were obtained, in particular, on Salisbury Plain, on Dartmoor, and in the neighbourhood of Nottingham. Examination of the curve shows,

* APPLETON and BARNETT, 'Nature,' Vol. 115, p. 333, 1925.

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however, that the readings obtained on these sites are not appreciably nearer to the mean curve than some taken in what at first sight appeared to be far less satisfactory positions.

In one particular case a reading was taken just at the top of a steep drop, in what was practically a Devonshire lane, the hedges on the side banks being above the level of the top of the coil and the lower wires below the surrounding ground level, but the reading was quite normal. The question of long-distance telegraph wires is a more open one, as it is conceivable that their effect may be dependent on the constants of the circuits, in which case immunity from trouble is merely a matter of luck.

All readings were taken at a distance of not less than 200 yards from such wires.

The results are shown in Fig. 5, and the small sketch map (Fig. 6) gives an indication

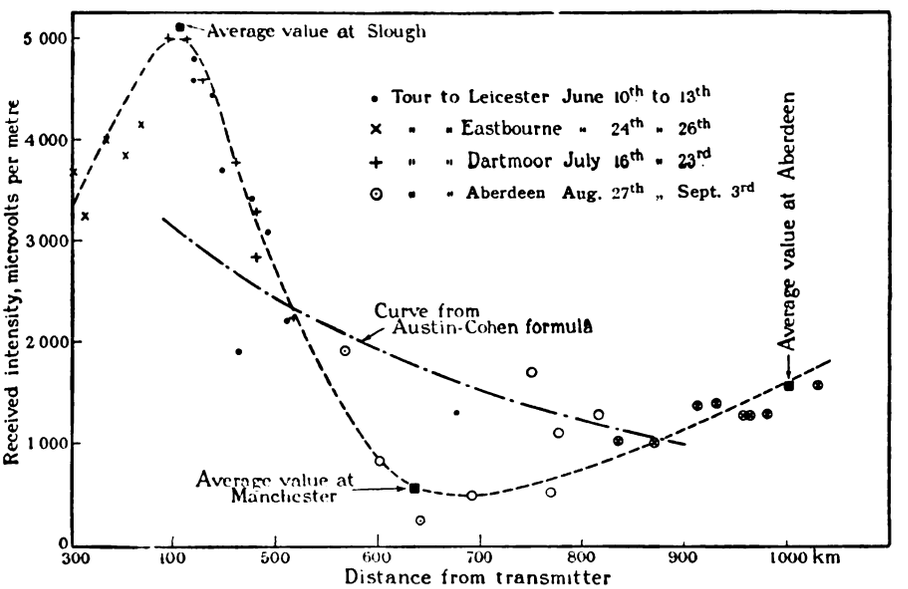

FIG. 5. Intensity-distance Curve during Summer Months taken on St. Assise (FT; 14,350 m.) in 1925 with a Portable Set.

of the points at which observations were taken, the difference in the bearing of FT at the extreme points being 42°. After allowing for all the factors mentioned above, the curves appear to be very definite and to be entirely free from anything in the nature of a steady decay, the dotted line being the theoretical value as calculated from the Austin-Cohen formula. This curve is given, as there is reason to think that, whilst the authors of it fully appreciate the limitations under which it is applicable, some subsequent writers have invested it with an accuracy and validity which were never originally claimed for it, and seem entirely unjustified. The curves seem to justify very definitely the claim for the existence of an interference phenomenon, and, owing to the greater number of points on them, allow conclusions to be drawn with a much higher degree of certainty than when observations were confined to the fixed stations.

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General Inferences from these Results.

In view of the fact that hardly any two of the points at which readings were taken lay exactly on the same great circle through the transmitter, it seems unlikely that these effects are confined to any particular direction. Also, the various paths contained very different proportions of sea and land, which again goes to confirm the view expressed

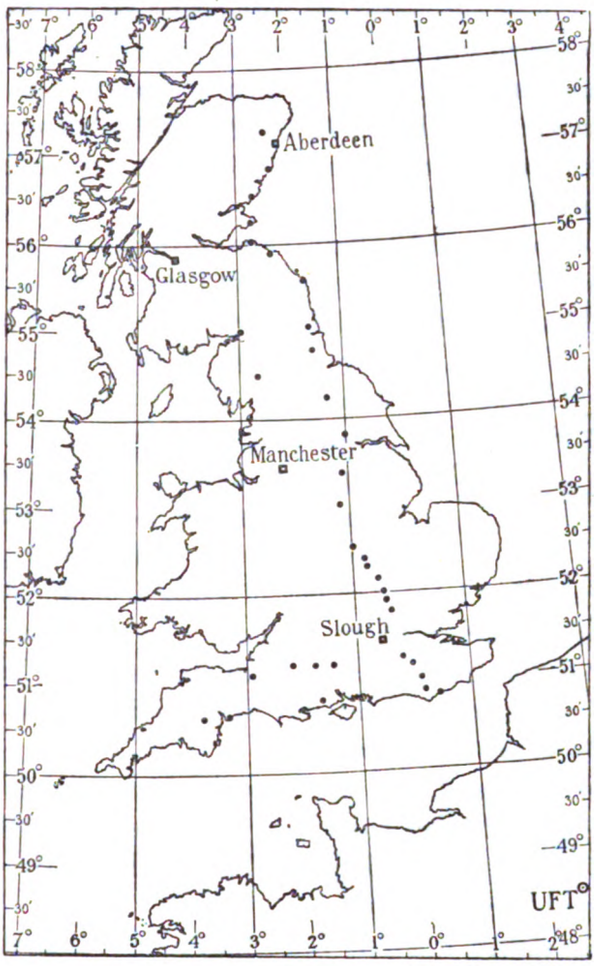

FIG. 6.

earlier in the paper, that for these wave-lengths the nature of the path is of minor importance.

An interference phenomenon demands two waves, so that it is necessary to give a short discussion of the means by which such two waves could arise. In some of SOMMERFELD's original work it is shown that the velocity of propagation of a "surface" wave over a semi-conductive layer is slightly different from that of a "free" wave in space. At first sight this suggests a possible explanation of the phase-changes necessary to produce an interference effect, but calculation shows that by the time a sufficient phase-

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change had occurred due to this cause, the " surface " wave would have been attenuated to a negligible intensity. It seems, therefore, that the explanation must be found rather in difference of path length than in velocity of propagation, and this leads at once to the idea of the upper layer. The existence of such a layer has been suggested from the earliest times, and forms the basis of most of the mathematical work on the subject; but opinion on this point is by no means unanimous, and it has been claimed that all observed phenomena can be accounted for without its assistance.

It is therefore proposed to assume as a starting point the existence of such a layer, but with its characteristics unspecified, and then to show that all the results obtained in the paper can be accounted for if certain definite constants are attributed to it. The question, by its very nature, does not permit of direct experiment, but can only be settled by the accumulation of evidence.

Now if on the surface of the earth (Fig. 7) one wave follows the arc AB between the two stations, whilst another is reflected at a point C on the upper surface, it is easy to calculate the phase-change due to the difference of the paths AB and ACB. At this stage of the work it is impossible to give a decision as to the actual physical nature of the layer, its constitution, or its sharpness of definition, the layer in Fig. 7 being treated as the imaginary sharp layer required to produce the same effect. Consequently, it cannot be said definitely whether the effect is strictly one of refraction or reflection, or, as seems likely, something of an intermediate nature. In all that follows, the word " refraction " is used in this indeterminate sense and must not be more strictly interpreted. Many more experiments under varying conditions are required before this further question can be answered in detail.

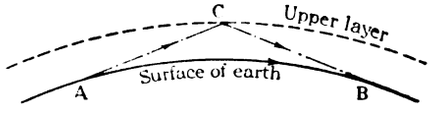

FIG. 7. Diagram Illustrating Wave Paths.

Returning, however, to Fig. 7, the distances at which maxima and minima should occur for various heights of layer have been calculated for the wave-length of FT ($\lambda = 14{,}350$ m.) and are plotted in Fig. 8. The full lines give the positions of the maxima, and the dotted lines the minima.

With regard to these curves, it is very important to note that the actual positions of the maxima and minima are independent of the attenuation of either of the waves, a fact which is of great value, as it eliminates some of the most uncertain of the unknown factors. The actual intensities at these points are, however, entirely dependent on the attenuations, so that the interference phenomena will exhibit all degrees of intensity. At short distances, where the surface wave is strong and the refracted wave weak due to the small angle of incidence, the maxima and minima will follow in rapid succession as the distance varies; but their actual excursions will probably be so small as not to be definitely separable from the irregular momentary and daily variations. As the distance from the transmitter increases, the effects will become more and more marked and, as appears from Fig. 5, an almost total extinction may be obtained; whilst at

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greater distances, due to the large attenuation of the surface wave, the effect will again diminish, if by this time it has not totally disappeared, as the last minimum will of course be at the point at which the two paths differ by half a wave-length. Thus, to exhibit this effect to the best possible advantage, the distances from the transmitter must be carefully chosen or it may be overlooked.

Now if Fig. 5 be compared with Fig. 8, it will be seen that the conditions are fairly well satisfied if the equivalent height of the layer be taken as 75 km., which should give a maximum at 430 km. and a minimum at 640 km., with no intermediate value.

Returning for the moment to Fig. 4, if its figures are compared with graphs of the same type as Fig. 8, but drawn for the appropriate wave-length, it will be found that, for the

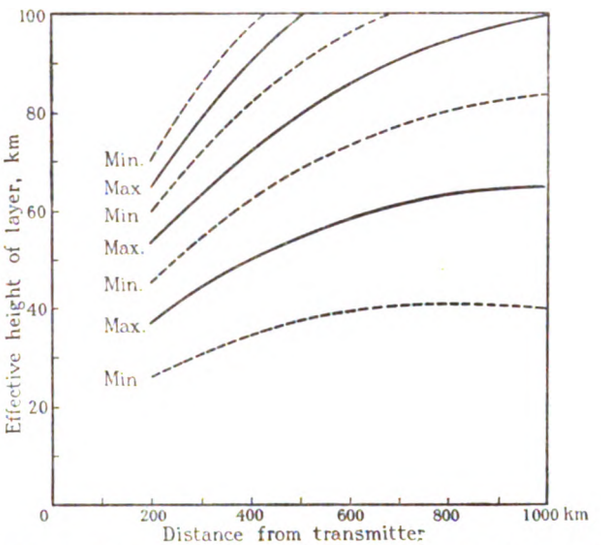

FIG. 8. Chart of Position of Interference Phenomena for $\lambda = 14{\cdot}35$ km.

September reading, a layer at a height of 72 km. should give a minimum at a distance of 450 km. and a maximum at 800 km. These correspond very fairly to Fig. 4, after making allowance for the fact that with only four points of observation the positions of the maxima and minima cannot be determined to any high degree of accuracy. The November reading in Fig. 4, on the other hand, suggests a height of the order of 85 km. It thus appears that, at any rate up to this date, all the major elaborate variations observed are explicable on the assumption of a refracting layer at a height of about 75 km. during the summer months, rising somewhat abruptly during October to a height of about 90 km.

This does not, of course, definitely prove the existence of such a layer, as any other theory which could account for the observed results would be equally acceptable. All that can be said at present is that it offers what appears to be the simplest explanation, which, at the same time, covers all the most marked observed phenomena.

Of the actual physical nature of the layer, present results are not sufficient to give any

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details. General conceptions are somewhat adverse to the idea of a sharply defined surface of separation, though it must be remembered that sharpness would only be a relative term, being in fact a function of the wave-length ; so that the degree of sharpness may be entirely different for long and short waves, with a resulting difference of observed phenomena. If, however, the suggestions put forward by ECCLES* or LARMOR† are accepted, the practical result is that the "effective height" would be different for different wave-lengths. The range of wave-lengths used in this paper is not sufficient to settle this point without special experiments designed to illustrate it, but it is hoped to undertake these in the future.

Seasonal Variations.

So far, results have been chiefly confined to the summer period. The next points arising are :—

(i) Comparison of results obtained during the summer and winter months.

(ii) Comparison of results during the same period in successive years.

The abrupt transition at the end of October appears to mark a distinct line between these two periods, and in this connexion also, the curve given in Fig. 2 is of special importance. It is obtained from the results published periodically by Prof. MESNY in ' L'Onde Electrique,' and its significance lies in the fact that the time of transmission (1955 G.M.T.) is close to the sunset period. Actually it is about the time of sunset at Bordeaux on the longest day, and at Paris in the middle of July. The first rise in the curve evidently corresponds to the normal sunset change, which is discussed in more detail below, but the striking point is that, after a comparatively steady period following this, another abrupt change occurs at exactly the same time as on the stations whose transmissions are in daylight throughout the year. As by this time of year sunset has occurred, both at Meudon and Bordeaux, several hours before the reading is taken, it seems impossible to argue that the winter change is merely a reversion to normal night conditions. The two may be related, and probably are, but they do not seem to be identical.

For the daylight stations, the results appear to group themselves into two sections, that from May to September being marked by its relative stability, and that from October to May both by average values differing largely from the summer ones, and also by a very pronounced day-to-day instability.

Also, it has been shown that the marked change at the end of October is consistent with a rise in height of the layer from 75 to 90 km., so that it seems natural to suggest that some, at any rate, of the winter variations are due to the same cause. Unfortunately, the large day-to-day variations referred to above cause great difficulty in

* ECCLES, ' Proc. Roy. Soc.,' A, Vol. 87, p. 79, 1912.

† LARMOR, ' Phil. Mag.,' Vol. 48, p. 1025, 1924.

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applying the system of weekly averages, which under these circumstances do not appear to possess much physical value. For this purpose a daily analysis of the results had been undertaken, in order to see if even the daily variations will allow of quantitative explanation, but the preliminary arithmetical work is very heavy and it is not possible to say yet whether the analysis will be successful.

One point in connexion with the layer appears to be very striking. It is clear from Fig. 8 that a change in height of only about 10 km. may be sufficient to convert the signal at a given point from a maximum to a minimum. Now the readings during the summer show that the day-to-day variations at any given point are often only a few per cent., and that even the weekly average for the same week in successive years may not differ by more than this amount. Consequently, although it appears that the layer undergoes a nightly change, it must go through a cyclic change of extraordinary constancy over periods measured by weeks, if not by months. It may be added that it is admittedly somewhat premature to draw conclusions as to yearly variations from observations extending over so short a period. Time alone can settle this question, but it is certainly striking that, as far as results go at present, there appear to be very marked signs of an annual repetition.

Sunset Effects.

A completely separate series of experiments suggested by my assistant, Mr. NAISMITH, have also been made. Continuous watch kept at Slough, during the summer months, on St. Assise, from about one hour before sunset until two hours after, showed that during the summer period the intensity went through a very marked cycle of change, which was repeated daily with only very minor modifications. Similar observations were therefore organised at the other three permanent stations, and the results showed that whilst the same thing was true at every station, the actual curves obtained had entirely distinctive and different shapes. These are given in Fig. 9 for FT at Slough, Manchester and Aberdeen, and in Fig. 11 for GKC (Northolt) at Manchester and Aberdeen. Fig. 10 is also interesting, as showing the curves obtained at the same time by two independent stations 24 km. apart. Neither of the observers knew that the other was working, and the results obtained have been plotted without correction, so that it would appear that the agreement between the various sets of apparatus is good. It will be seen at once from these curves that the effect can no longer be classed under the general heading of "night increase," but that it must be, although perfectly systematic, of a much more complex nature. Now it is generally held by the supporters of an upper-layer theory that, with the removal of sunlight, the effective height of the layer increases.

Following out this idea, if the curves in Fig. 9 are compared with the graphs in Fig. 8, the variations at all stations are explicable on the basis of a rise in effective height from 75 km. to 90 km., bearing in mind the fact which has been referred to above, that a minimum is not necessarily a zero.

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Another interesting, though tentative, calculation can be made from these results. From Fig. 9 it is possible to determine the actual values of the maximum or minimum attained at each station, and sometimes both, depending on the range of the curves.

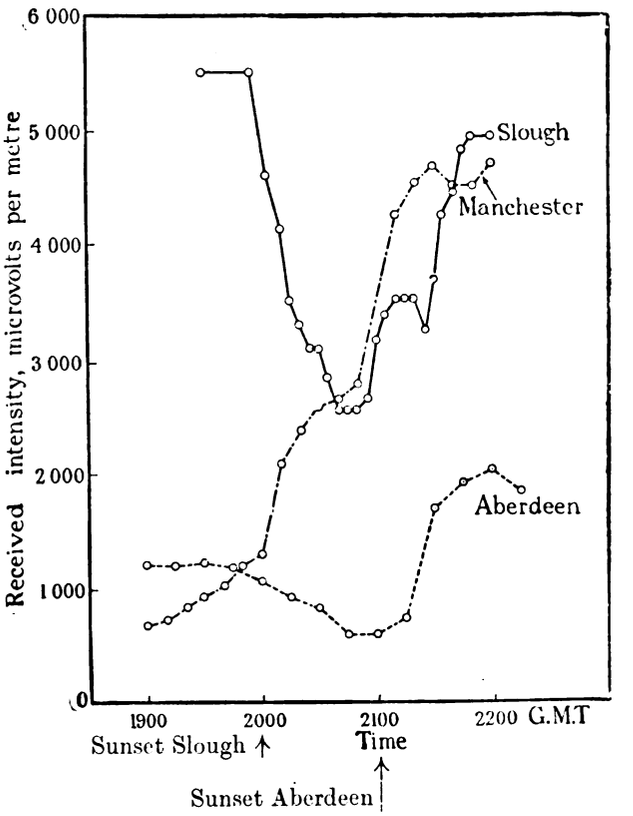

FIG. 9. Sunset Run on St. Assise (FT ; 14,350 m.), July 21, 1925.

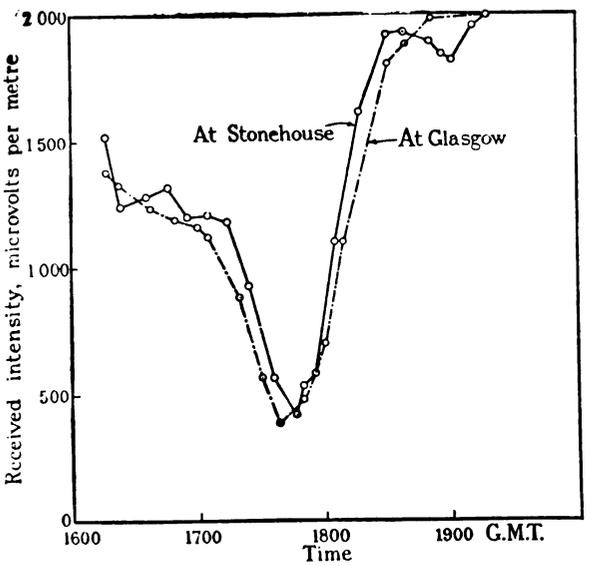

FIG. 10. Sunset Run on FT (St. Assise) on October 12, 1925. Stations 24 km. apart.

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The intensity of the diffracted wave can also be calculated from the formula given by WATSON* and by VAN DER POL,† and hence the intensity of the reflected wave is known.

From this the equivalent reflecting power of the layer can be deduced, on the assumption that the intervening air is a perfect dielectric.

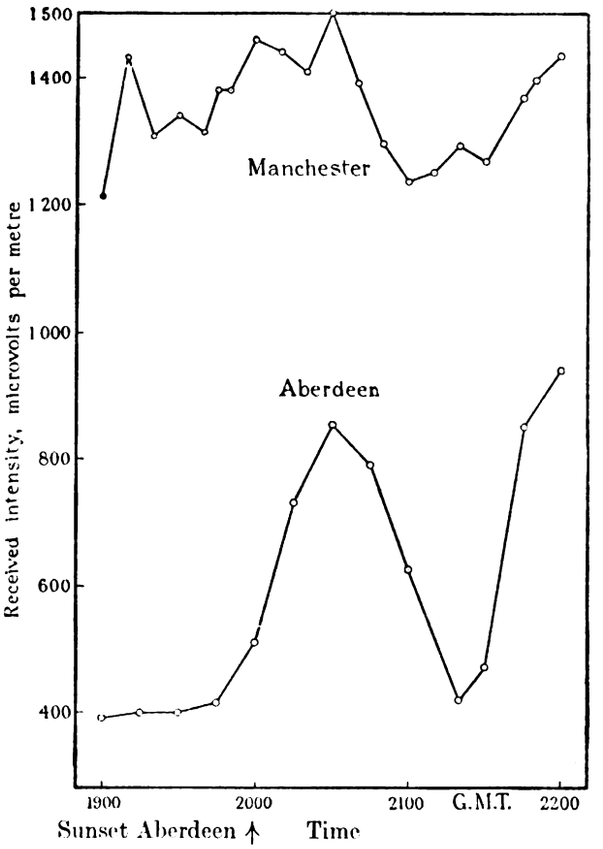

FIG. 11. Sunset Run on Northolt (GKC ; 6,950 m.), July 30, 1925.

The figures actually obtained are shown in Table II.

A corresponding figure derived from the Northolt results is 0·6 for the reflection coefficient, but it must be remembered that the idea of working out these results only occurred some time later. It has consequently been necessary to assume the nominal value of aerial current, and if this was not the actual value when the observation was taken the reflection coefficient would be correspondingly different ; in fact, further work suggests that the Northolt figure should also be of the order 0·4.

These results are interesting, especially as showing the rapid falling off of reflecting power for larger angles of incidence, which is not altogether unexpected, but like many

* WATSON, ' Proc. Roy. Soc.,' A, Vol. 95, p. 83, 1919.

† VAN DER POL, ' Phil. Mag.,' Vol. 38, p. 365, 1919.

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TABLE II.

	St. Assise (FT ; $\lambda = 14{,}350$ m.).			
	At Slough.	At Manchester.	At. Glasgow.	At Aberdeen.
(i) Maximum value from curve	(1)	(1)	(1)	2,000
(ii) Minimum value from curve	2,500	600	400	600
(iii) Calculated surface wave	4,000	1,980	1,000	900
(iv) Reflected wave	750	1,300	700	750
(v) Calculated maximum value	5,500	4,600	2,800	2,100
(vi) Reflection coefficient (Note 2)	0·16	0·42	0·37	0·42
(vii) Angle of incidence on upper layer ...	68·5°	75·5°	78·5°	79·5°

Note (1)—It is not always possible to determine these values reliably from the curves. For instance, in the case of a V-shaped curve such as that for Slough (Fig. 9) there is no definite evidence of a point of inflexion having been reached either at the beginning or end. The missing values have therefore been calculated in (v) in order to see whether they are of the right order, and they appear to be in most cases very close to figures which have at times been obtained (*cf.* Fig. 5 for the maximum value at Slough).

Note (2)—The reflecting power of the ground has been taken as unity, as Dr. SMITH-ROSE's results (see Paper 16, p. 362) have shown that this is not far from the actual figure.

other figures in the paper they must only be taken as provisional, and subject to further verification by the method of daily analysis referred to previously.

Conclusions from the Results.

Whilst many of the ideas suggested in this paper cannot at present be regarded as much more than guesses at a possible state of affairs, it may be as well to collect and re-state some of the definite facts that rest on experimental evidence, and which must form a basis for any theory which can cover them.

(1) Long-wave signal intensities over medium distances ($< 1{,}200$ km.) remain extremely steady over the period May to September, during daylight hours.

(2) Their distribution does not correspond to the Austin-Cohen formula for anything less than yearly averages, in which case the distance is not large (see Table III) and the law of decay is not uniform.

(3) During the sunset period each station goes through a distinct and individual cycle of change, which is repeated daily.

(4) Towards the end of October all observed stations showed simultaneous and large departures from their summer values. This is the beginning of a period of marked day-to-day instability, which persists until the following May, by which time all stations have settled down to their normal summer values.

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(5) Careful plotting of a day-time intensity-distance graph gave an undulating curve, showing, under certain conditions, practical extinction at a definite distance.

(6) In addition to the slow periodic changes, there appear to be more rapid fluctuations superimposed. During the summer months they are comparatively small, but assume big intensities during the winter. There is not yet evidence to decide whether these are due to fluctuations of layer height too small and rapid to be definitely detected by present methods, or whether they are due to some totally separate cause.

TABLE III.

	UA.		GBL.	
	Yearly average.	Austin-Cohen value.	Yearly average.	Austin-Cohen value.
Slough	1,880	1,660	5,560	5,650
Manchester	1,070	1,040	2,100	2,260
Glasgow	690	650	820	770
Aberdeen	370	530	510	590

Extension to Long Distances and Short Waves.

The actual results in this paper are definitely confined to medium distances and long waves, as the deliberate idea throughout has been to investigate phenomena rather than results ; but at the same time it is interesting to inquire how the ideas can be applied to these other cases.

(*a*) *Long Distances.*—The interference phenomena discussed above have a definite upper limit of distance for two reasons. The first is that, on the assumption that they are caused between a refracted and a diffracted wave, the attenuation of the latter is so large that at great distances its effect is negligible. Secondly, these effects cease when the difference in path length for the two waves becomes less than half a wave-length ; and this, together with the curvature of the earth, places a definite geometrical limit on the possible distance, there being also a maximum range for a first reflection with a given height of layer.

The idea of multiple reflection cannot, however, be excluded (*cf.* Captain ROUND and Mr. ECKERSLEY*) and it is conceivable that some effect of the following nature may take place.

The signal from a distant station would consist of the vector sum of a series of multiple reflections. The least number of reflections possible is given by dividing the total distance by the maximum range of a single reflection leaving the earth tangentially

* ROUND and ECKERSLEY, 'J. Inst. Elec. Eng.,' Vol. 63, p. 995, 1925.

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(in the case of a layer 75 km. high this figure is 1,950) and taking the next integer above this.

Theoretically all multiple refractions of a higher order than this would be present, forming an infinite series, but actually, owing to the loss at each refraction and the gradually increasing angle of incidence, their relative attenuations would be rapid, and probably only two or three would survive to a measurable extent. These might even show interference phenomena among themselves, though at large distances the effect would be so sensitive to variations in the layer too small to produce any effect at short distances that it would for practical purposes be incalculable.

(*b*) *Short Waves.*—There is danger in applying these results directly to wave-lengths below 1,000 metres, for, whilst the presence of such a suggested upper layer would be a very important factor in short-wave propagation, the difference in frequency is so great that there is no doubt but that the actual observed phenomena would be entirely different.

Conclusion.

In conclusion, it is again well to lay emphasis on the fact that, whilst the results given in the paper are reasonably definite, the conclusions drawn from them must not be treated as more than suggestions with varying degrees of validity. The primary object has been to deduce from somewhat general observations definite systematic points requiring further investigation, and it is not until confirmation, or the reverse, is obtained that any opinion can be expressed as to the degree of actuality of the suggested phenomena.

This work was carried out for the Radio Research Board of the Department of Scientific and Industrial Research. I must here express my gratitude to the observers at my out-stations, Dr. FYVIE at Aberdeen, Mr. HOYLE at Manchester, and Messrs. DALGLEISH and NAISMITH at Glasgow. To them has fallen the part of taking the routine and somewhat monotonous observations without which none of the results could have been obtained.

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XXI.

THE POLARISATION OF RADIO WAVES

BY

J. HOLLINGWORTH, M.A., D.Sc., M.I.E.E.,

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ABSTRACT.

The paper deals with the propagation of long radio waves, waves longer than 10,000 m., and with distances between 400 and 1,000 km. (though results below 400 km., suggesting entirely different conditions of propagation, are indicated briefly). Present results show that : (*a*) during the hours of darkness, long wave intensities are moderately steady but show considerable abnormal polarisation ; (*b*) at a time before sunrise corresponding to the sun's position of 7° below the horizon at a point on the earth's surface mid-way between transmitter and receiver, an abrupt change lasting about 12 minutes occurs, during which the polarisation becomes much less ; (*c*) a similar effect, but with increase of polarisation, occurs at sunset, its period being sometimes rather less definite and slightly longer ; (*d*) at sunrise the value of the reflection coefficient, which has remained at about its night value throughout period (*b*), falls gradually to its day value, which in summer is usually about 30 per cent. of its night value (but appears to be zero occasionally), and during this period the remainder of the abnormal polarisation disappears ; (*e*) the evidence points to a much smaller change in equivalent height of the layer than has been assumed up to the present ; it also suggests that a considerable proportion of the loss of energy in the indirect ray may occur below the actively refracting part of the layer rather than in it. The author also inclines to the idea that no real mechanical rising of the layer occurs, but merely a change in its electrical and physical state. The abrupt changes in the rotation of the plane of polarisation are shown in the curves ; the actual amount of rotation occurring varies daily, and a discussion of it is reserved for another paper.

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THE POLARISATION OF RADIO WAVES.

By J. HOLLINGWORTH, M.A., D.Sc., M.I.E.E., Sometime Assistant in the Electricity Department.

[Reprinted from the 'Proceedings of the Royal Society,' Series A, Vol. 119, 1928.]

Introductory.

The following paper is primarily an attempt to study by experimental methods the problem of the propagation of long radio waves between two points on the surface of the earth. In the early days of radio-telegraphy the propagation formulæ in general use were admittedly empirical, being derived from the simple expressions for propagation over a plane conducting surface, and modified by factors deduced from the statistical mean of large numbers of observations. While such formulæ served their immediate practical purpose with considerable success, it was evident from their origin that they could give but little physical insight into the actual process of propagation. Further experiments were carried out with a view to increasing the data available, but their general result was to indicate that the process was far too complex to be expressed in any reasonably simple mathematical form, and the investigation then gradually assumed the more theoretical form to which attention is being directed at present.

Now it was suggested in the early days that it was extremely probable that some form of electrical layer in the upper atmosphere was responsible for many of the phenomena; and in view of the more recent work by later observers, especially Smith-Rose and Appleton, it may be taken that this view has been definitely confirmed; it is indeed now tacitly accepted by the great majority of investigators. Starting from this point, mathematical investigations have been developed, including those of Eccles,* Larmor,† and Nichols and Schelleng,‡ most of which take into account in addition the effect of the earth's magnetic field. But in the formulæ obtained in this way appear certain constants of the upper atmosphere to which at present no definite numerical values can be assigned, owing to our limited knowledge of the electrical and physical conditions at heights greater than 8 to 10 kilometres.

Owing to the presence of these unknowns, the most important of which is the variation of ionic density with height, these formulæ cannot be used for the direct calculation of received radio intensities; in fact the most promising line of attack seems to be in the

* Eccles, 'Proc. Roy. Soc.,' A, Vol. 87, p. 79, 1912.

† Larmor, 'Phil. Mag.,' Vol. 48, p. 1025, 1924.

‡ Nichols and Schelleng, 'Bell System Tech. J.,' Vol. 4, p. 215, 1925.

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other direction, namely, the use of the experimentally measured values as a means of checking the validity of the general assumptions, and also, if possible, for the determination of some of the above-mentioned constants. It is with this end in view that the experiments here described have been planned and carried out, and while they only touch the fringe of the subject, which becomes more and more complex the further it is investigated, they seem to have yielded a few definite facts which must be taken into account in any theory of propagation.

Method.

The absolute measurement of received radio intensities is a subject which has probably made less progress than most other radio measurements, owing to the technical difficulties involved. In the first case the received powers are extremely small, of the order of 10^{-12} watt, so that considerable amplification is essential, with its attendant problems of instability and parasitic disturbances. Secondly, the actual production for purposes of calibration of extremely small high-frequency E.M.F.'s of accurately known value has not yet been placed on a standardised basis, and almost every experimenter uses a different method, while the methods themselves have as yet been mutually compared only to a small extent. The earliest observations of the intensity of received radio signals consisted almost entirely of simple measurements of the electric field intensity from very distant transmitting stations, usually at least 2,000 km. away. This, of course, gave immediate practical information in its most useful form ; but such results were not always of the highest theoretical value, for which the experiments should be designed to produce the physical facts under observation in their most marked and easily measurable form.

It was therefore necessary for the present purpose to make very considerable modifications in apparatus and routine, though both of these have been matters of gradual growth rather than of abrupt transition from the older ways. Early experiments carried out by the author on the old lines, though made at considerably shorter distances (300 to 1,000 km.), had shown conclusively that several independent variables were involved, and that, therefore, adequate information could not possibly be obtained from single simple intensity measurements. In this connexion it is important to point out that work of this nature differs from most other physical work in that the independent variables, even if their existence is fully understood, are incapable of control and independent variation, since they are really of the nature of cosmic phenomena. Moreover, on long wave-lengths at any rate, it is impossible to vary transmitter conditions as regards wave-length, situation, and time of sending, to the extent that pure theory demands ; the only method is to make the best use possible of the standard commercial transmissions. Progress is therefore slow and laborious. The origin of the experiments in this paper dates back to January, 1924, when four simple intensity measuring sets were installed under the auspices of the Radio Research Board of the

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Department of Scientific and Industrial Research at Slough, Manchester, Aberdeen and Glasgow (later transferred to Exeter), to work to a routine supplied to them by the author. The analysis of the results received from these stations over a period of about two years provided further evidence in favour of an interference effect between a surface wave and one returned from the upper layer, and this was subsequently confirmed by a special experiment; but during the winter period (November to April), marked though systematic irregularities occurred, which were clearly inexplicable on the basis of the simple assumptions which had so far been found sufficient. About this time, papers by ECKERSLEY* and in particular SMITH-ROSE and BARFIELD† suggested that the errors in direction-finding on distant stations, which also become more pronounced annually during the winter period, were due to abnormal polarisation of the downcoming wave. Efforts to collate these errors with the above intensity variations led, however, to no definite results; and it was NAISMITH's discovery in May, 1926, that the daily cycle of intensity variation which occurred regularly on certain stations during the hours around sunset was invariably accompanied by an equally distinct and definite cycle of bearing variation, which first established the connexion in a measurable form. Experiments were immediately put in hand to try and put this on a quantitative basis, and preliminary results were communicated in a letter to 'Nature.'‡ The results of these experiments, analysed on lines shortly to be described, are given in Fig. 1, but their numerical accuracy is obviously low, although comparison with later results (Fig. 4) shows that their main features are correct.

At this stage it was felt that further qualitative evidence was unnecessary, and efforts were henceforward concentrated on the improvement of experimental method and accuracy.

Theory of the Method.

Since this work is not concerned at the moment with the actual physical mechanism by which the effects are produced at the upper layer, it will be sufficient to start from the general assumption that the downcoming wave is abnormally polarised, *i.e.*, that it contains a horizontal component of electric force, and that it is the interference between this complex wave and the surface wave which produces the effects under consideration.

The necessary equations will therefore contain six unknowns :—

(I) H_0, the magnetic intensity of the surface wave.

(II) H_1, the magnetic intensity of the normal component of the downcoming wave.

* T. L. ECKERSLEY, 'Radio Review,' Vol. 2, p. 60, 1921.

† SMITH-ROSE and BARFIELD, 'N.P.L. Collected Researches,' Vol. 21, Paper 17, p. 365 (or 'Proc. Roy. Soc.,' A, Vol. 116, p. 682, 1927.

‡ J. HOLLINGWORTH, 'Nature,' Vol. 118, p. 409, 1926.

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(III) H_1^1, the component of magnetic intensity in the vertical plane containing the direction of propagation.

(IV) θ_1, the phase angle between H_0 and H_1.

(V) θ_1^1, the phase angle between H_0 and H_1^1.

(VI) The " equivalent height " of the layer.

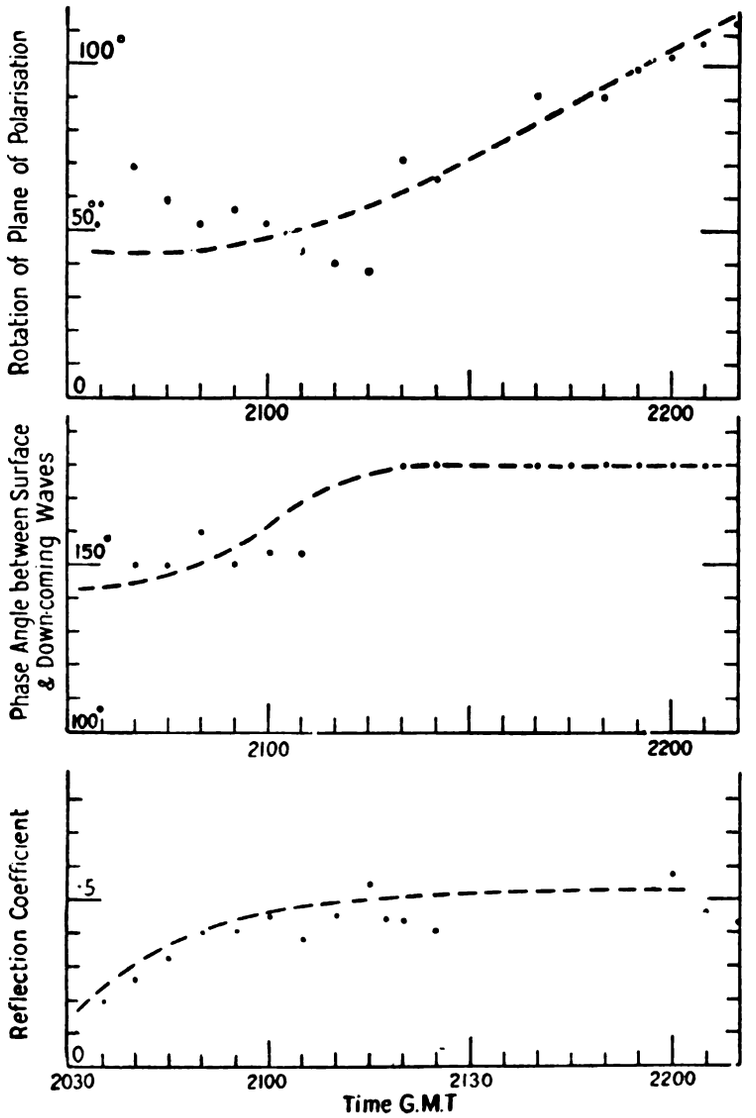

FIG. 1. Analysis of Sunset Run on St. Assise (14,350 m.). Sunset 2009 G.M.T.
(This was the first ever obtained and was taken with the old apparatus on June 28, 1926.)

H_0, H_1 and H_1^1 are all assumed to be of simple harmonic form in time. Of these, H_0 can be calculated from one of the standard diffraction formulæ, and (VI), the height of the layer, is roughly known from APPLETON's* work and from previous work of the

* APPLETON and BARNETT, ' Nature,' Vol. 115, p. 333, 1925.

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author. It is also assumed, following the work of SMITH-ROSE,* that at these wave-lengths the surface of the earth may be assumed to be a perfect reflector.

Four unknowns being left, it will therefore in general be necessary to take four distinct measurements for each determination.

Now with this wave system the resultant horizontal magnetic intensity at right angles to the plane of propagation is

$$H_0 + 2H_1 \text{ (vector addition)},$$

and the resultant horizontal intensity in the plane of propagation is $2H_1^1 \sin \theta$, where θ is the angle which the direction of propagation of the downcoming wave makes with the surface of the earth. Hence the E.M.F. induced in a vertical coil oriented at an angle ϕ to the geographical direction of the arriving signal will be proportional to :—

$$(H_0 + 2H_1) \cos \phi + 2H_1^1 \sin \theta \sin \phi \text{ (vector addition)}.$$

Examination of this equation will show that by variation of ϕ, the only parameter under experimental control, three, and only three, separate equations can be obtained. APPLETON† obtained a fourth on his shorter waves by the artifice of wave-length variation, but since the percentage change required is proportional to the wave-length, this is impossible in the present case on practical grounds. In theory this fourth equation can also be obtained by the use of a vertical aerial in conjunction with the coil ; but unfortunately this method is only applicable when the transmitter is at a comparatively short distance (< 200 km.) from the receiver, since the new equation is the same as one of the others except for a factor $\cos \theta$ in one term, which, if θ is small, only differs from unity by an amount comparable with the instrumental accuracy of the whole apparatus.

Therefore, to obtain a determinate solution from three observations, it is clearly necessary to make a further assumption. The one taken is that the downcoming wave is not elliptically but plane polarised, though with its plane of polarisation rotated from the normal position. (This must be taken to include the possibility of elliptic polarisation the eccentricity of which cannot be detected owing to the limits of instrumental accuracy.) While on purely general grounds this assumption is quite unjustifiable, the results so far obtained yield a considerable amount of indirect and circumstantial evidence in its favour. In order, however, to develop the arguments on this point, it is first necessary to describe in some detail the method of obtaining the solution and the particular form in which the three readings are now taken.

On the assumption of plane polarisation, if H^1 be the magnetic intensity of the downcoming wave, η the angle of rotation of its plane of polarisation from the normal position,

* SMITH-ROSE and BARFIELD, ' N.P.L. Collected Researches,' Vol. 21, Paper 15, p. 317 (or ' Proc. Roy. Soc.,' A, Vol. 107, p. 587, 1925).

† APPLETON, ' Proc. Roy. Soc.,' A, Vol. 117, p. 576, 1928.

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the equations for the E.M.F.'s induced in two vertical coils set 45° on either side of the geographical direction are

$$\left.\begin{aligned}\sqrt{2}E_1 k &= H_0 + 2H_1 \cos\eta + 2H_1 \sin\eta \sin\theta \\ \sqrt{2}E_2 k &= H_0 + 2H_1 \cos\eta - 2H_1 \sin\eta \sin\theta\end{aligned}\right\}$$

(vector addition), where k is an instrumental constant.

If, therefore, we can construct a triangle (Fig. 2) OAB such that

$$OA = \sqrt{2}E_1 k \quad \text{and} \quad OB = \sqrt{2}E_2 k,$$

and join O to C, the mid point of AB, it follows that

$$OC = H_0 + 2H_1 \cos\eta$$
$$CA = CB = 2H_1 \sin\eta \sin\theta.$$

Hence AB gives the phase position of H_1, and therefore by describing a circle with centre O and radius H_0 to cut AB in D and E, either CD or CE gives $2H_1 \cos\eta$ in magnitude and phase. (In any particular case the choice between CE or CD can be settled on inspection by a method which is somewhat elaborate to explain and not essential to the present argument.)

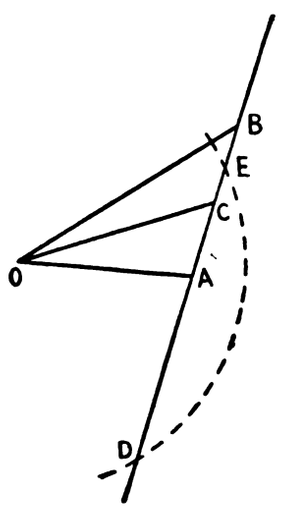

FIG. 2.

It is therefore evident that in every case the solution is obtained by the construction of the triangle OAB.

Now following out SMITH-ROSE'S idea, since the primary symptoms of abnormal polarisation are the absence of a sharp minimum on a direction-finding coil and an error in the apparent bearing, the first three observations selected were :—

(i) the intensity in the normal (maximum) position of the coil,
(ii) the intensity in a direction at right angles to this,
(iii) the bearing error,

these being taken as rapidly after one another as possible. The results in Fig. 1 were actually obtained from such observations ; but it was soon found that their accuracy was low. In the first place the intensity (ii) is often very much smaller than intensity (i), involving either a loss of valuable time while the apparatus is readjusted, or else a reading of low accuracy for observation (ii), affected severely as it is by jamming owing to its low intensity. Also, when abnormal polarisation is large, the minimum becomes very flat, resulting in an extremely low accuracy of observation (iii). This method was therefore replaced by one in which three intensity observations at different orientations

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were made. Again, however, the accuracy was low, partly due to the fact that the apparatus then in use was not sufficiently refined for the purpose, but also owing to a purely geometrical factor. For in this case we have to construct the triangle OAB given OA, OB and OC; and when these are nearly equal and AB is small compared with them (as often happens in practice), the shape of the triangle is excessively sensitive to small changes in these lengths. It was, therefore, finally decided to construct completely new apparatus in which the three quantities measured were the E.M.F.'s in two equal coils at right angles to one another set at 45° on either side of the geographical direction, and, as a third observation, the phase angle between these E.M.F.'s, *i.e.*, the sides OA, OB and the angle AOB in Fig. 2. The set also incorporated several refinements not found in the earlier ones. Since the completion of this set in June, 1927, it has been found to give a very great improvement in results, and all further deductions have been based on these three observations.

It is now possible to discuss the evidence in favour of plane polarisation. In the first place, while from the purely mathematical standpoint we have only three equations for four unknowns, there is the further physical criterion that any solution must be real and physically possible, and probably possess reasonable continuity in time. In the present case this requires that the ratio of the actual intensity of the downcoming wave to its theoretical value calculated on the assumptions that the layer is a perfect reflector and the intervening medium a perfect dielectric, must always be less than unity. (Incidentally this ratio is in future referred to as the " equivalent reflection coefficient " of the layer.)

Any assumption which consistently gives results satisfying this condition is at any rate worthy of further investigation. There is, however, more direct evidence. Examination of the observations taken in a considerable number of sunrise and sunset runs shows that the phase angle between the E.M.F.'s in the two coils as a general rule changes very much more slowly than their intensities. In one particular case, in the course of 6 minutes the E.M.F.'s in the two coils fell to half their initial values, while the phase angle between them remained constant at 35°. Examination of the resulting vector diagram shows that if in this case the polarisation had been circular, it would have been necessary for the two vectors H_1 and H_1^1 in this period to go through two distinct and elaborate intensity cycles depending not only on themselves but also on their momentary phase and intensity ratios to H_0, actually reversing their direction of rotation relative to H_0 during the period.

On the other hand, the result can at once be interpreted numerically, on the assumption of plane polarisation, by a change of 50° in the angle of rotation of this plane, without even a change in the value of the reflection coefficient. Of these two alternatives the second is so very much simpler that there seems every justification for adopting it. The particular case quoted was rather a marked example; but the comparatively slow change of phase is a definite characteristic of these runs, in fact many cases have occurred in which the whole sunrise cycle has been completed with the E.M.F.'s in the

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two coils in phase throughout, and some results are definitely insoluble unless the vectors H_1 and H_1^1 are in phase or nearly so. Hence in the remainder of this paper, plane polarisation will be assumed, but it should be noted that there is no intention of extending it by analogy over any range of wave-lengths not covered by these actual experiments.

From the diagram in Fig. 2 the quantities η, H_1, and the phase angle between H_0 and H_1 can easily be calculated.

Description of the Apparatus.

Briefly, the apparatus consists of two coils about 5 feet square mounted at right angles to one another about a common vertical axis, the whole being free to rotate. Each coil is wound with 54 turns of wire, and is separately tuned and its circuit completed through one of the fixed coils of a radio-goniometer. This is merely two small coaxial coils mounted at right angles to one another in the exact position of zero mutual inductance, and another smaller coil, also coaxial with them, but free to rotate about its axis and fitted with a scale and pointer. The whole of the goniometer is heavily screened in a sheet-iron box. The search coil is connected to a third tuned circuit, also screened, the condenser terminals of which are joined to the grid and filament of the first valve of an amplifier. The amplifier is of the resistance-capacity type, which, though of slightly lower efficiency than some other forms, will work over a considerable wave-length range without re-adjustment. The output from the last valve is connected to a crystal in series with a small string galvanometer fitted with a microscope and eye-piece scale, on which the actual deflections caused by the incoming Morse signals can be read directly. For calibration, the input terminals of the amplifier are connected to a source of known and adjustable E.M.F. of the same frequency as the incoming signal. From this reading and the known circuit constants of the set, the incoming E.M.F. can be calculated. For determination of the phase angle use is made of the fact that, if the two E.M.F.'s in the main coils are in phase, the currents induced in them will give rise to a linear resultant in the goniometer, so that, by rotating the search coil to a position at right angles to this, a clear sharp zero will be obtained on the string galvanometer. If, however, they are not in phase, their currents will give rise to an elliptical field in the goniometer, and no absolute minimum will be obtainable on the search coil. But by slightly de-tuning one of the coils, so as to alter the phase relation between the E.M.F. and the current in it, the sharp minimum can again be obtained, and the reading of the amount of de-tuning required (which is done by means of a calibrated condenser) together with the knowledge of the circuit constants, gives the value of the phase angle between the two E.M.F.'s. This has been found to work very satisfactorily except in the two extreme cases, when the original phase difference is nearly 90° (which hardly ever happens), or when one of the E.M.F.'s is so much smaller than the other that the exact position of de-tuning balance is uncertain. This has occurred occasionally, but in this case the abnormal

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polarisation is so great that fairly accurate results can be obtained even from rough observations.

Experimental Results.

After some consideration and several experiments it was finally decided to confine the observations for the time being to a single transmitting station. The reason for this was that it soon became clear that there were very distinct day-to-day variations; so that to begin by observations on several stations would merely have increased the number of independent variables without adding any generalisation of any value.

Moreover, when abnormal polarisation is present, unambiguous values of the quantities measured cannot be obtained from a single observation. For the effect of a 180° change of phase between H_0 and H_1 with unaltered polarisation is identical with that of a 180° change of rotation of plane of polarisation with unaltered phase, although the intermediate stages are altogether different. Hence it is strictly necessary to take a series of observations beginning or ending with a time at which the state of polarisation is roughly known. Now all experience shows that the rotation is zero during the day (in summer at any rate), and so for definite results a complete sunset or sunrise cycle is required. It is, however, possible to dispense with this occasionally after the general characteristics of a particular transmission have become known by experience. Thus in general it is impossible to take simultaneous comparative observations on more than one station without duplication of apparatus.

The transmission first adopted was that from St. Assise (near Paris) on 14,350 metres. This has the advantage that it is at a distance from Slough which exhibits the phenomena under consideration in a very marked way. Also its transmission is very good and steady, and it is working most hours of the day and night. The author is under a particular debt of gratitude to the Compagnie Radio-France, who have kindly supplied him with a great deal of information without which he would have had great difficulty in making the necessary calculations.

The first observations were made during a period covering the solar eclipse of June 29, 1927, as it was thought that useful information might be obtained from the combination of these and the eclipse observations. As an example of the results obtained, Fig. 3 gives in graphical form the actual intensities measured on the two coils during sunset on the day preceding the eclipse, from which it will be seen that the variations are considerable and reasonably continuous; in fact it seems quite certain that there are no large fluctuations with periods measurable in seconds. The string in the galvanometer indicating the Morse signals comes to a clear and definite position for each dot and dash (except in the case of high-speed automatic sending), and it is this position which slowly fluctuates as shown in the curves.

The analysis of these curves is given in Fig. 4, and many sunrise and sunset curves have been obtained since that date, of which Fig. 5 is a recent specimen. Individually

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they show certain variations, for a full examination of which it will be necessary to accumulate more data covering all seasons of the year. Such observations, however, have to be started at least one hour before sunrise, and sometimes continued till after sunset. Owing to the use of a null method for the phase measurement, it cannot be

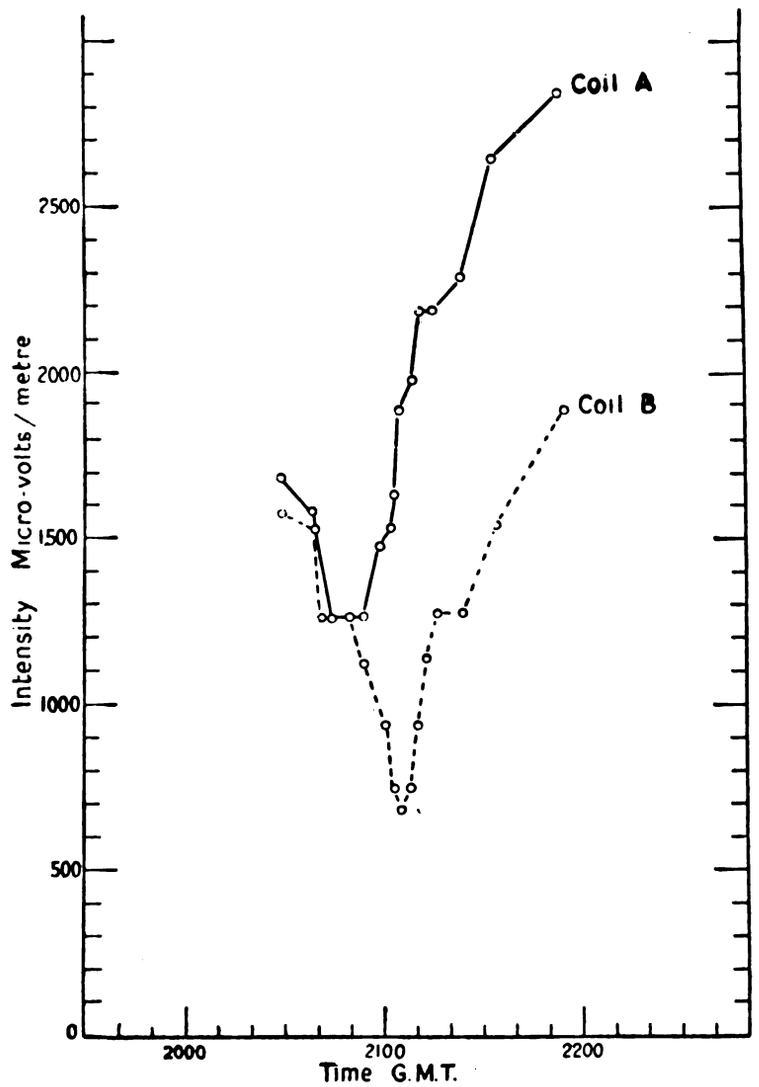

FIG. 3. Curves showing the Actual Readings obtained on the Measuring Set, from which the Results in Fig. 4 are calculated.

done automatically, and the attendance of an experimenter is required continuously, with the result that physical limitations lead to a comparatively slow accumulation of results. But all those so far obtained show a certain number of distinct and well-marked similarities, and it is only with the discussion of these that the present paper is concerned.

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General Deductions.

The three outstanding physical quantities arising out of the results can be grouped as follows :—

(*a*) Reflection coefficient.
(*b*) Rotation of plane of polarisation.
(*c*) Phase relation between surface and downcoming waves.

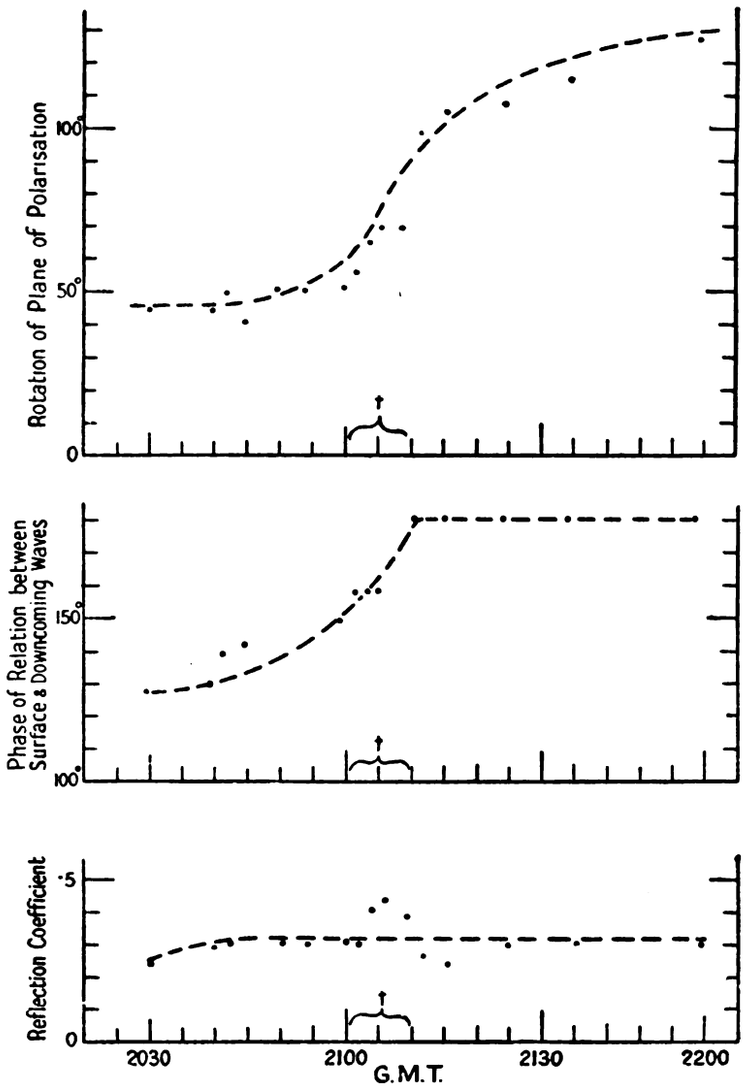

FIG. 4. Analysis of Sunset Run on St. Assise, taken at Slough, June 28, 1927.

(*a*) *Reflection Coefficient.*—The reflection coefficient has been defined as the ratio of the actual downcoming intensity to the value it would have if reflection at the layer were perfect and the air a perfect dielectric. It therefore includes absorption losses in the intervening medium as well as actual reflection losses. One of the most striking of its features is its great constancy in spite of the large observed variations from which

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it is calculated. For example, the two following observations taken entirely at random from a single run both gave the same value (0·5) for the reflection coefficient :—

Intensity in coil A.	Intensity in coil B.	Phase angle.
800	1100	+26°
3300	2200	−17°

Similar cases are of continual occurrence, and this in itself is taken as a further, though indirect, justification of the assumption of plane polarisation. The most usual night value is from 0·3 to 0·4, with occasional ranges from 0·2 to 0·6. The day value (which is somewhat difficult to determine) is usually 0·1 to 0·15, though there are signs that it is occasionally zero. No attempts have yet been made to correlate it with other phenomena, and at present its value seems to be entirely a matter of chance.

Another feature is that the rise from the daytime value at sunset and the corresponding fall at sunrise take place about the times of sunset and sunrise on the earth's surface, or even before and after them respectively. The exact times of these changes and the daytime value itself are rather difficult to determine accurately, as, when the reflection coefficient and abnormal polarisation are both small, high accuracy is required to obtain soluble equations. The fact, however, remains definite, and the full implication of it will be discussed later.

(*b*) *Rotation of the Plane of Polarisation.*—All the curves, specimens of which are given in Fig. 5, show distinctly that the change in the rotation of the plane of polarisation is in two parts, each fairly abrupt, and that the first of these parts is over in from 10 to 12 minutes. The exact time of its commencement can often be determined more accurately at sunrise than at sunset, since the reflection coefficient is usually higher and the effect on the intensities more marked. Under favourable conditions the commencement of the rapid change (see Fig. 7) can be determined to half a minute, and so sunset runs are now seldom taken. From this curve it will be seen that while there is a gradual drop for 15 minutes or so preceding the critical time (0547), at this instant there is an unmistakable change in rate of decrease in intensity. Observations were actually taken at one-minute intervals throughout the run ; but prior to 0547 they are not all shown on the diagram, as they lay uniformly on the curve drawn. It will be noticed that this change is shown much more prominently in Fig. 7 than in Fig. 5. The reason for this is that in Fig. 5 the angle of rotation is derived from its tangent, and when the angle is near 90° its tangent changes much more rapidly than the angle itself. (In all these diagrams except Fig. 7 the actual curves shown are only intended as rough indications of the forms obtained.) In connexion with this time of commencement there are some interesting deductions. At the time of writing (October, 1927) the effect is occurring about 45 minutes before sunrise at the point midway between transmitter and receiver. On examining previous records for June, the time interval is found to be 55 to 60 minutes, the lower accuracy being merely due to the fact that at that time its importance was not fully appreciated and its time of incidence was not so

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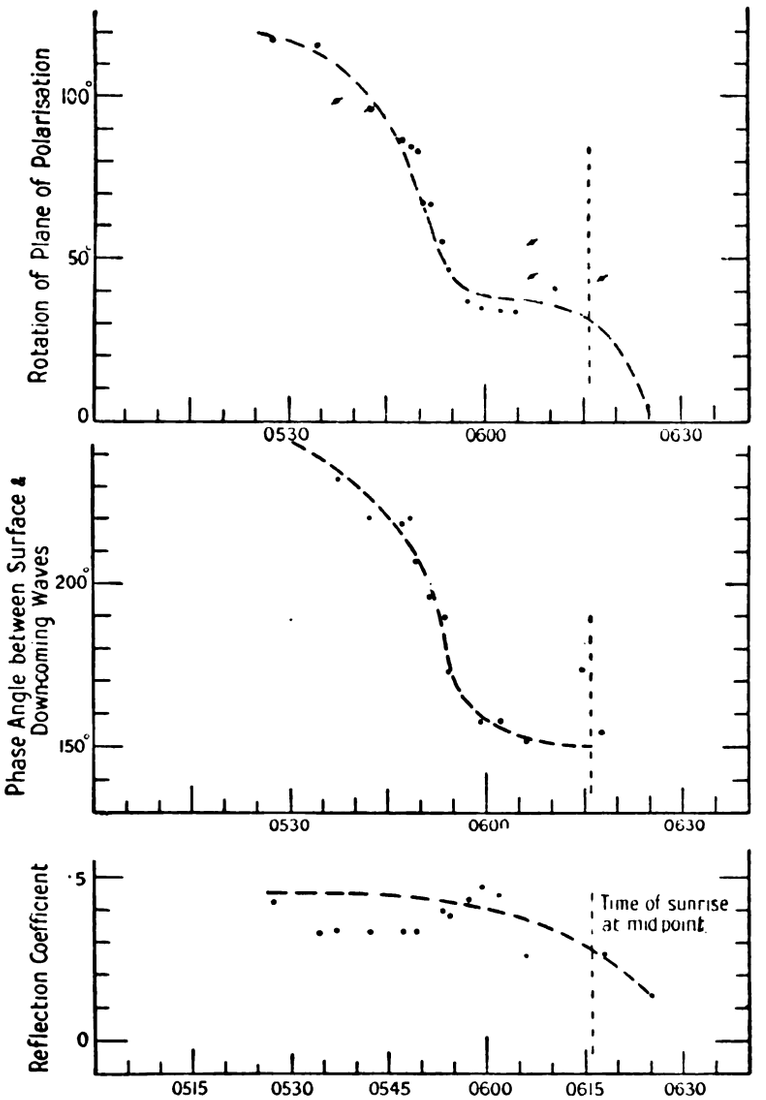

FIG. 5. Analysis of Sunset Run on St. Assise taken at Slough, October 25, 1927.

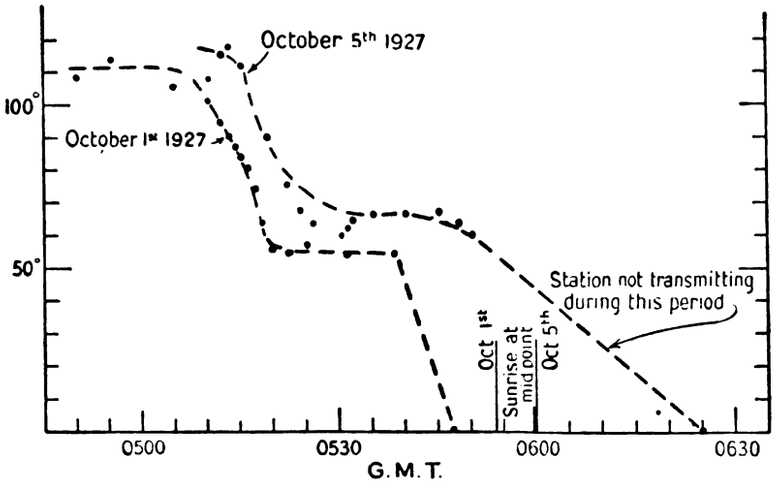

FIG. 6. Typical Polarisation Curves for Sunset Period.

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accurately noted. By calculation from the actual readings it is found that, owing to the change in the sun's declination between these periods, they all correspond to a position of the sun 7° below the horizon at the mid-point referred to above. We can thus assume that the ionisation of the upper layer by sunlight starts when the sun is 7° below the horizon at the point on the earth's surface vertically below. Now this means that the sun is just rising (neglecting refraction) at a point in this vertical line

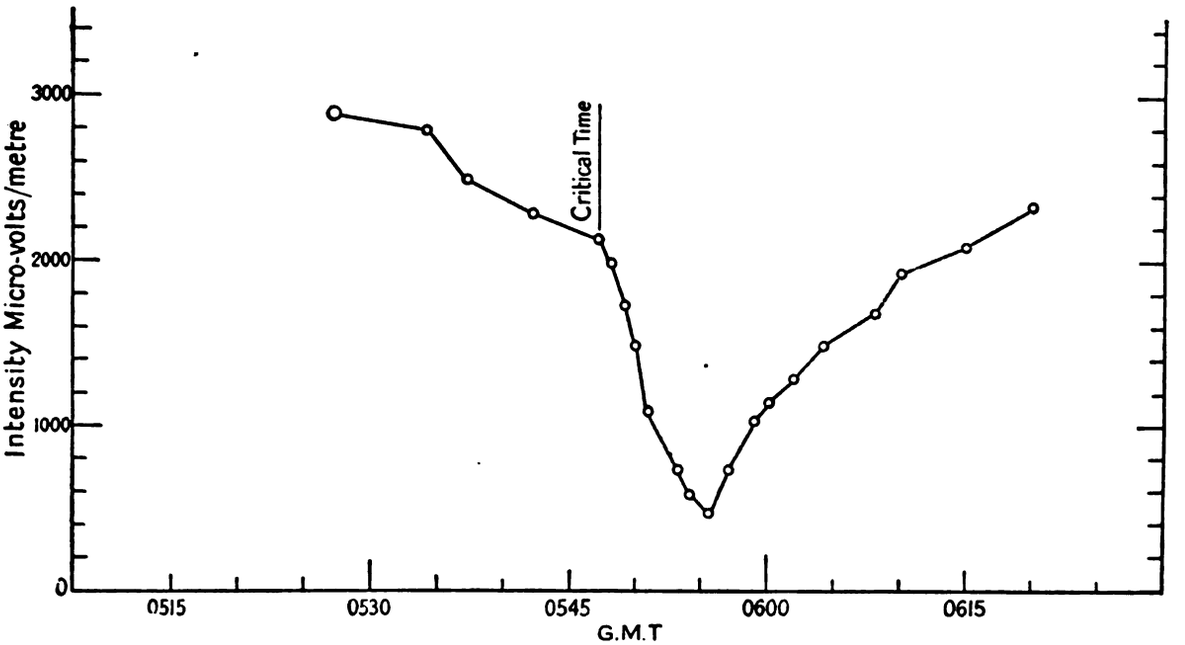

FIG. 7. Graph of Actual Readings taken on October 25, 1927, showing critical period at 0547 G.M.T.

47 km. above the earth's surface. This fixes a lower limit for the height of the layer. The actual height must be greater for the following reason. Such a ray, being tangential to the earth's surface, would pass twice through the thickness of the earth's atmosphere, and so would have its ionising power completely destroyed. It is not until it has risen sufficiently above the lower atmosphere to retain this power that it can begin to have any effect on the layer. So little is known about the absorbing power of the upper atmosphere that this cannot be allowed for, and it seems more reasonable to argue that, as other purely radio experiments give the height of the layer as 75–85 km., the absorbing power of the atmosphere for ultra-violet radiations persists up to a height of 30 to 40 km. above the earth's surface, after which it ceases fairly abruptly. Further experience has suggested that this absorption of the sun's ionising radiation by a medium through which it has passed may be an important factor in several radio phenomena.

During the first change the rotation usually drops to about 30 per cent. of its night value and remains constant for a time until the second abrupt change occurs, when it again drops, this time to zero (summer period). The exact time of this second change is more difficult to determine, for the reasons given above, but it appears to be about sunrise on the earth's surface at the mid-point. Now this time, the sun's rays being

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then tangential to the layer, is, of course, a transition period between the time when the layer is illuminated from below and that (normal daytime) when it is illuminated from above. It is conceivable that these two methods of illumination cause different ionic gradients, and so account for the change ; but of course this has, at present at any rate, no proof.

The actual amount of rotation occurring varies daily, and a discussion of it is reserved for another paper.

Observations continued throughout the night have shown that after the rapid sunset changes there may be slight further changes for an hour or so, after which a steady state is reached which persists all night. Actually in the middle of the night the intensities seem to be almost as steady as in the day, but of course the values are usually very different. (This applies primarily to the summer months, extended data for the winter period not having been obtained at present.)

It should be noted that there is always a considerable difference in time between the commencement of the abnormal polarisation and of the change of reflection coefficient referred to in the preceding section. While the former appears abruptly on an average about 45 minutes after sunset and before sunrise, the latter is less definite, but is certainly much nearer to the actual times of sunrise and sunset on the earth's surface at the mid-point. Now at these latter times the layer itself is in full sunlight, and has been or will remain so for at least another half-hour, so that there is a distinct suggestion that an appreciable amount of the energy loss of the reflected wave occurs in the medium between the ground and the active layer, rather than in the layer itself. This of course is no new idea, but rather a confirmation of an old one by a new method. If this is the case the expression " reflection coefficient " will have a very limited and special meaning. For this reason it was, on its introduction, specially referred to as the " equivalent reflection coefficient," and defined on p. 437 in such a way as to include the actual reflection coefficient, for it will in this case be not only a function of the layer constants but will also contain a distance attenuation factor as well.

(*c*) *Phase Relation between Surface and Downcoming Waves.*—One of the most unexpected results of these analyses was the small change in the phase relation between these two waves during the sunset and sunrise periods due to change of equivalent path length. Previously the " V " curve obtained for the intensity during the sunset cycle had been explained as entirely due to this, and a consequent change of layer height of 10 to 15 km. inferred therefrom.

It now appears, however, that this effect was largely caused by abnormal polarisation, and that the actual change of equivalent path length is very much smaller, corresponding only to a change of 1·5 to 2·0 km. at the utmost in the most marked cases. Frequently throughout the whole of a sunrise cycle no visible phase change occurs, and in one particular case recently the E.M.F.'s in the two coils remained in phase for nearly 24 hours, though during that period each of them went through cycles of intensity of the order of 5 to 1 in magnitude and also changed their relative values by considerable amounts.

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It has always been felt that the idea of equivalent height had but little physical reality, and in view of these results one is inclined to argue that no real mechanical rising of the layer occurs, but merely a change in its electrical and physical state, and that the phase change effects are secondary effects due to the alteration of the value for the reduced path of the ray in the layer as its electrical constants change. Examination of the results so far obtained does not seem to show any connexion between this phase change and the other factors in the transmission, but for this purpose a larger number of observations will probably be required than are at present available.

Before finally summing up it may be well to give a short discussion of probable errors and their effect on the results.

Quantitative radio work suffers from many limitations. On the purely technical side the instrumental accuracy is inevitably much lower at these high frequencies than it is at those in commercial use, and in addition, when working on ordinary station routines (as is almost essential in long-wave work), neither the time nor the nature of the transmission is under the observer's control. It frequently happens that in the middle of a vital series of observations the station shuts down. No organisation could remedy this, as experiments are so frequently dictated by the needs of the moment. In addition to this, all the calculations involve the use of vector diagrams, and cases are bound to occur in which the triangles involved are of such degenerate form that the accuracy of solution is low.

Tests have shown that under favourable conditions relative measurements can be obtained to an accuracy of 2 per cent., and it is on these that the shape of the vector triangle usually depends. Due to both internal and external causes, absolute accuracy is lower, probably 5 per cent. to 10 per cent., but in these calculations the chief effect is to change the reflection coefficient by a similar percentage, and at times to make a somewhat greater change in the angle of rotation. Neither of these changes has any serious effect on the present work.

Summary.

It will be seen that the present paper is rather of an introductory nature to a subject which has as yet received but little systematic study. Routine observations are now proceeding in order to obtain a greater number of data, and to study certain special effects. Even from the present results, however, certain general characteristics have shown themselves in an indisputable manner, and it is with the discussion of these that the paper is chiefly concerned. In summary form its main conclusions are given below.

(*a*) During the hours of darkness long wave intensities are moderately steady, but show considerable abnormal polarisation.

(*b*) At a time before sunrise corresponding to the sun's position of 7° below the horizon at a point on the earth's surface mid-way between the transmitter and receiver, an abrupt change lasting about 12 minutes occurs, during which the polarisation becomes much less.

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(*c*) A similar effect with increase of abnormal polarisation occurs at sunset, but its period is sometimes rather less definite and slightly longer.

(*d*) At sunrise the value of the reflection coefficient, which has remained at about its night value throughout the period (*b*), falls gradually to its day value, which in summer is usually about 30 per cent. of its night value (but appears to be zero occasionally), and during this period the remainder of the abnormal polarisation disappears.

(*e*) The evidence points to a much smaller change in equivalent height of the layer than has been assumed up to the present; it also suggests that a considerable proportion of the loss of energy in the indirect ray may occur below the actively refracting part of the layer rather than in it.

Note.—The general features of these deductions are unaffected by the particular form of polarisation assumed, but it must again be emphasised that they only apply to long waves ($>$ 10,000 metres) and to distances of transmission between 400 and 1,000 km.

General Observations.

In view of the increasing use and importance of short waves it may be thought that the time is now past for the study of long-wave phenomena. While it must be admitted that no theory or study of propagation is complete which does not deal with short waves, it must be pointed out that the critical study of the subject by other than statistical methods has not yet advanced very far. It seems advisable, therefore, to study the subject under the conditions in which it is most amenable to quantitative analysis, rather than necessarily under the practical conditions of everyday use; and for this purpose there is no doubt that long waves possess very great advantages, as the instrumental problems which arise are very much less, and experience had shown that the variations which occur are slower and less erratic. Logically, therefore, it seems that the long wave problems should be studied in detail first, the extension to shorter waves following later.

Below 400 km. distance the conditions of propagation appear to be entirely different, and are being made the subject of a separate investigation. Their salient characteristics appear to be (1) reflection coefficient usually zero, but may be as high as 0·4; (2) whenever a refracted wave is received, by day or night, it is abnormally polarised, and no marked change occurs during the sunrise and sunset periods; (3) the few measurements so far obtained suggest an effective vertical height of penetration of about 150 km.

With regard to Prof. APPLETON'S recent letter to 'Nature'* as to the existence of another higher layer operative in the middle of the night, no definite evidence has yet been obtained. Although the height of the present layer assumed (80 km.) is too low for direct determination by the aerial-coil method for stations at the distances apart

* APPLETON, 'Nature,' Vol. 120, p. 330, 1927.

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employed in the present paper, this would not be the case for a height of 150 km. Tests of this method have been made during the night, but unfortunately at this time the usual state of polarisation of the signals is a rotation of about 90°, in which case the critical term in the equation is nearly zero and a solution is practically unobtainable.

On the other hand, if the signals penetrate to such a layer during the night and fall back to the normal one during the day, one would expect enormous variations at the period of change, if not almost a discontinuity, and in none of the observations made so far has any such effect been noted, although the signal is under absolutely continuous observation during the periods of watch.

At distances above 1,000 km. the scientific side of the problem becomes much more complex. Multiple reflection may occur, and signs of it at this distance of 1,000 km. have recently been detected and are now being investigated. Abnormal polarisation may also be present, though it cannot in general be detected or measured owing to the absence of an appreciable ground wave and the high angle of incidence of the refracted ray, which make the ordinary polarisation measuring methods unworkable. It seems likely, however, that it should always be considered as a possible factor in any theory of long distance propagation, as, if present, small changes in it may have very large effect on the signal intensity, a wave with its electric vector horizontal having no effect whatever on normal receiving apparatus.

The work described was carried out as part of the programme of the Radio Research Board of the Department of Scientific and Industrial Research. Finally, I must express my sincere gratitude to my colleague, Mr. NAISMITH, of the Radio Research Station, Slough, both for his suggestion and carrying out of the original experiment which led to the work being undertaken, and also for his share in the construction of the apparatus and in making the observations, involving as this does protracted and delicate observations over periods of several hours at all times of day and night.

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